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सर्वश्रेष्ठ परीक्षा पथ-प्रदर्शक

B.Sc. First Year (Certificate Course) Semester-I

MATHEMATICS

Differential Calculus & Integral Calculus

SYLLABUS

PART-A : DIFFERENTIAL CALCULUS

UNIT-I Introduction to Indian ancient Mathematics and Mathematicians should be included under Continuous Internal Evaluation (CIE).(Appendix)

Neighbourhood of a point, bounded above sets, bounded below sets, Bounded Sets, Unbounded sets, open sets/intervals, closed sets/intervals, Limit points of a set, Isolated points, Limit, continuity and differentiability of function of single variable, Cauchy's definition, Uniform continuity, boundedness theorem, Intermediate value theorem, extreme value theorem, Darboux's intermediate value theorem for derivatives and Chain rule.

UNIT-II Rolle's theorem, Lagrange and Cauchy Mean value theorems, Taylor's theorem with various forms of remainders, Successive differentiation, Leibnitz theorem, Maclaurin's and Taylor's series. Partial differentiation, Euler's theorem on homogeneous function.

UNIT-III Tangent and Normal, Asymptotes, Curvature, Envelops and evolutes, Tests for concavity and convexity, Points of inflexion, Multiple points, Parametric representation of curves and tracing of parametric curves, Tracing of curves in Cartesian and Polar forms.

UNIT-IV Definition of a sequence, theorems on limits of sequences, bounded and monotonic sequences, Cauchy's convergence criterion, Cauchy sequence, limit superior and limit inferior of a sequence, subsequence, Series of non-negative terms, convergence and divergence, Comparison tests, Cauchy's integral test, Ratio tests, Root test, Raabe's logarithmic test, de Morgan and Bertrand's tests, alternating series, Leibnitz's theorem, absolute and conditional convergence.

PART-B : INTEGRAL CALCULUS

UNIT-V Concept of partition of interval, Properties of Partitions, Riemann integral, Criterion of Riemann Integrability of a function, Integrability of continuous and monotonic functions, Fundamental theorem of integral calculus, Mean value theorems of integral calculus. Differentiation under the sign of Integration.

UNIT-VI Improper integrals, their classification and convergence, Comparison test, μ-test, Abel's test, Dirichlet's test, quotient test, Beta and Gamma functions.

UNIT-VII Rectification, Volumes and Surfaces of Solid of revolution, Pappus theorem, Multiple integrals, change of order of double integration, Dirichlet's theorem, Liouville's theorem for multiple integrals.

UNIT-VIII Vector Differentiation, Gradient, Divergence and Curl, Normal on a surface, Directional Derivative, Vector Integration, Statements of Theorems, of Gauss, Green & Stokes, only without proof, Applications of these theorems for evaluation of double and triple integrals.

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Part-A : Differential Calculus

UNIT I

SECTION-A VERY SHORT ANSWER TYPE QUESTIONS

Q.1. Find the supremum and infimum of the set $S = [x \in z : x^2 \leq 25]$.

Ans. Since $S = [x \in z : x^2 \leq 25] = [-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5]$

Since S is a finite subset of R, the smallest member of S is –5, which is a lower bound of S and hence infimum of S is –5. Similarly 5 is the supremum of S.

Q.2. The null set ϕ is neither bounded below or above, nor unbounded.

Ans. Since, there is no member in ϕ, we can not check whether a given real number can be a bound for ϕ or not. Thus, bounds for ϕ do not exists. On the other hand we can as well say that every real number is a lower or upper bound for there is no member in ϕ which does not satisfy the required property of bounds.

Q.3. Find the limit points of the closed interval [0, 1].

Ans. Let $A = [0,1]$

Then, in a similar manner as in Ex. 1.

We have $D[(0,1]) = [0,1]$

Q.4. Define the term 'Supremum'.

Ans. If s is an upper bound of a subset S of R and any real number less than s is not an upper bound of S, then s is called the least upper bound (*l.u.b*) or supremum (sup) of S.

Q.5. What do you mean by unbounded set of real numbers ?

Ans. A subset S of R is said to be unbounded if it is not bounded above or not bounded below.

Q.6. Define the term 'Bounded intervals'.

Ans. If a and b are any real numbers that $b > a$, then the intervals $]a, b[, [a, b], [a, b[$ and $]a, b]$ are called bounded intervals because each of them is a bounded subset of R.

Q.7. Evaluate $\lim_{x \to 0} \left(x \sin \frac{1}{x} \right)$.

Ans. Let $f(x) = x \sin \frac{1}{x}$.

Then $\text{RHL} = f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} (0+h) \sin \left(\frac{1}{0+h} \right) = \lim_{h \to 0} h \sin \frac{1}{h}$

$= 0 \times$ a finite quantity lying between –1 and 1

$= 0.$ ...(1)

Also, $\text{LHL} = f(0-0) = \lim_{h \to 0} f(0-h)$

$$= \lim_{h \to 0} (0-h)\sin\left(\frac{1}{0-h}\right) = \lim_{h \to 0} h \sin\frac{1}{h}$$

$$= 0. \qquad \ldots(2)$$

Now, from (1) and (2) we find that RHL = LHL = 0.

Hence, $$\lim_{x \to 0}\left(x \sin\frac{1}{x}\right) = 0.$$

Q.8. What do you mean by unbounded intervals?

Ans. If a is any real number, then the intervals $]a, \infty[$, $[a, \infty[$, $]-\infty, a[$ and $]-\infty, a]$ are called unbounded intervals because each of them is not a bounded subset of *R*.

Q.9. Is it true that the null set ϕ is a *nbd* of each of its points?

Ans. Yes, the null set ϕ is a *nbd* of each of its points because there is no point at all in ϕ and so there is not point in ϕ of which it is not a *nbd*.

Q.10. Define the term 'Isolated point'.

Ans. A point $a \in A$ is said to be an isolated point of *A* if it is not a limit point of *A i.e.,* if there exists a *nbd* of a which contains no points of *A* other than a itself.

Q.11. Show that the function $f(x) = x|x|$ is differentiable at the origin.

Ans. Here, we have

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h|h| - 0}{h} = \lim_{h \to 0} h = 0$$

Now $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = \lim_{h \to 0} \frac{-h|h| - 0}{-h} = \lim_{h \to 0} h = 0$$

$\Rightarrow \quad Rf'(0) = Lf'(0).$

Hence, $f(x)$ is differentiable at $x = 0$.

Q.12. Show that $f(x) = \dfrac{x^2 - 1}{x - 1}$ is continuous for all values of x except $x = 1$.

Ans. If $x \neq 1$, then $f(x) = (x+1) =$ A polynomial

$\Rightarrow f(x)$ is continuous for all values of $x \neq 1$.

If $x = 1$, $f(x)$ is of the form $\frac{0}{0}$, which is not defined and so the function $f(x)$ is discontinuous at $x = 1$.

Q.13. Show that the function $f(x)$ is defined by $f(x) = \begin{cases} x^2, x \neq 1 \\ 2, x = 1 \end{cases}$ is discontinuous at $x = 1$.

Ans. Here the value of $f(x)$ at $x = 1$ is 2

$\Rightarrow \quad f(1) = 2.$

Now, $\text{RHL} = f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} (1+h)^2 = 1$

$$\text{LHL} = f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} (1-h)^2 = 1$$

Therefore, we have

$$f(1+0) = f(1-0) \neq f(1)$$

$\Rightarrow$ $f(x)$ is not continuous at $x = 1$.

Q.14. A function f is defined by

$$f(x) = x^p \cos(1/x), x \neq 0 \text{ and } f(0) = 0.$$

Find the differentiability at $x = 0$.

Ans. Let us suppose $p > 0$,

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{(h)^p \cos(1/h) - 0}{h}$$

$$= \lim_{h \to 0} h^{p-1} \cos 1/h \qquad \text{...(1)}$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{(-h)^p \cos(-1/h) - 0}{-h}$$

$$= \lim_{h \to 0} -(-1)^p h^{p-1} \cos(1/h) \qquad \text{...(2)}$$

Now if $f'(x)$ exists at $x = 0$, then we must have $Rf'(0) = Lf'(0)$ and this is possible only if $p - 1 >, 0$ *i.e.*, $p > 1$ which gives $Rf'(0) = Lf'(0)$. Hence in order that f is differentiable at $x = 0$, p must be greater than 1.

Q.15. A function f is defined as follows

$$f(x) = \begin{cases} 1 + x \text{ if } x \leq 2 \\ 5 - x \text{ if } x > 2. \end{cases}$$

Test the character of the function at $x = 2$ as regards its differentiability.

Ans. We have

$$Rf'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{5 - (2+h) - 3}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

and $$Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h \to 0} \frac{1 + (2-h) - 3}{-h} = \lim_{h \to 0} \frac{-h}{-h} = 1.$$

$\Rightarrow$ $Rf'(2) \neq Lf'(2)$.

Hence, the function $f(x)$ is not differentiable at $x = 2$.

SECTION-B SHORT ANSWER TYPE QUESTIONS

Q.1. Give examples to show that the union of an infinite collection of closed sets is not necessarily closed.

Ans. Let $F_n = [1/n, 1], n \in \mathbf{N}$.

Then each F_n is a closed set in **R** because each closed interval is a closed set.

Now $\cup \{F_n : n \in \mathbf{N}\}$

$$= \{1\} \cup [\frac{1}{2}, 1] \cup [\frac{1}{3}, 1] \cup [\frac{1}{4}, 1] \cup \ldots =]0, 1].$$

Since $]0, 1]$ is not a closed set in **R**, therefore it follows that the union of an infinite collection of closed sets is not necessarily closed.

As an other illustration consider the set **Q** of all rational numbers.

We can write $\mathbf{Q} = \cup \{r\}, r \in \mathbf{Q}$.

Since each singleton set in **R** is closed, therefore **Q** is expressed as an infinite union of closed sets. But we know that **Q** is not closed since its complement **Q**′ is not open. Hence it follows that an infinite union of closed sets in not necessarily closed.

Q.2. In the closed interval $[-1, 1]$ let f be defined by

$$f(x) = x^2 \sin(1/x^2) \text{ for } x \neq 0 \text{ and } f(0) = 0.$$

In the given interval (i) Is the function bounded? (ii) Is it continuous? (iii) Is it uniformly continuous ?

Ans. (i) If $x \in [-1, 1]$ and $x \neq 0$, we have

$$|f(x)| = |x^2 \sin(1/x^2)| = |x^2| \cdot |\sin(1/x^2)|$$
$$= |x|^2 \cdot |\sin(1/x^2)| \leq 1 \cdot 1 = 1.$$

$$[\because |\sin(1/x^2)| \leq 1 \text{ and } -1 \leq x \leq 1 \Rightarrow |x| \leq 1]$$

Also $f(0) = 0 \Rightarrow |f(0)| = 0 < 1$.

Thus $|f(x)| \leq 1, \forall x \in [-1, 1]$ also so f is bounded in $[-1, 1]$.

(ii) Let $c \in [-1, 1]$ and $c \neq 0$.

We have $\lim_{x \to c} f(x) = \lim_{x \to c} x^2 \sin \frac{1}{x^2} = c^2 \sin \frac{1}{c^2} = f(c)$.

$\therefore$ $f(x)$ is continuous at every point c of $[-1, 1]$ if $c \neq 0$.

Now to check the continuity of $f(x)$ at $x = 0$.

We have $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h), h > 0$

$$= \lim_{h \to 0} (-h)^2 \sin \left\{ \frac{1}{(-h)^2} \right\} = \lim_{h \to 0} h^2 \sin \frac{1}{h^2} = 0.$$

$$\left[\because \lim_{h \to 0} h^2 = 0 \text{ and } \left| \sin \frac{1}{h^2} \right| \leq 1 \text{ if } h \neq 0 \right]$$

Again $\quad f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)=\lim_{h\to 0} h^2 \sin\frac{1}{h^2}=0.$

Also $\quad f(0)=0.$

Since $f(0-0)=f(0)=f(0+0)$, therefore $f(x)$ is continuous at $x=0$.

Thus $f(x)$ is continuous at each point of $[-1,1]$ and so it is continuous in $[-1,1]$.

(iii) Since f is continuous in the closed interval $[-1,1]$, therefore it is also uniformly continuous in $[-1,1]$.

Q.3. *A, B* are sets such that $a\in A, b\in B \Rightarrow a<b$. Show that

$$\textbf{l.u.b.}\, A \le \textbf{g.l.b.}\, B.$$

Ans. Let l. u. b. $A=s$ and g. l. b. $B=t$.

To show that $\quad s\le t.$

Suppose if possible $\quad s>t.$

Since l.u.b. $A=s$ and $t<s$, therefore there exists $x\in A$ such that $x>t$.

Now g.l.b. $B=t$ and $x>t \Rightarrow$ there exists $y\in B$ such that $y<x$.

Thus there exists $x\in A$ and $y\in B$ such that $x>y$ which is against the hypothesis that $a\in A, b\in B \Rightarrow a<b$.

Hence our initial assumption $s>t$ is wrong and we must have $s\le t$.

Q.4. Find the limit points of the interval $]0,1[$.

Ans. Let $A=]0,1[$.

Now, firstly we shall show that every point of the closed interval [0, 1] is the limit point of A.

Let $p\in[0,1]$. Then for $\varepsilon>0$, the open interval $]p-\varepsilon, p+\varepsilon[$ must contains infinitely many points of A, therefore, it contains at least one point of A other than p.

$\Rightarrow$ p is the limit point of $]0,1[$.

Now, we shall show that no points, other than [0, 1] is the limit point of $]0,1[$ [*i.e.*, $p\notin[0,1]$, then p is not the limit point of $]0,1[$.

Let $\varepsilon>0$ be such that ε is less than the distance of the point p from each of the end points 0 and 1 of the closed interval [0, 1].

$\Rightarrow$ $\varepsilon<|p-0|$ and $\varepsilon<|p-1|$.

$\Rightarrow$ the open interval $]p-\varepsilon, p+\varepsilon[$ does not contain any point of the set A.

$\Rightarrow$ p is not the limit point of A.

Therefore, p is the limit point of $]0,1[$ if and only if $p\in[0,1]$.

Hence, $\quad \mathbf{D(]0,1[)=[0,1]}.$

Q.5. Evaluate $\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)$.

Ans. We have

$$f(x)=\frac{x^n-a^n}{x-a} \quad\Rightarrow\quad f(a+h)=\frac{(a+h)^n-a^n}{a+h-a}$$

$$=\frac{1}{h}\left[\left\{a^n+na^{n-1}.h+\frac{n(n-1)}{2!}a^{n-2}.h^2+\ldots\right\}-a^n\right]$$

Now, $\quad \text{RHL}=f(a+0)=\lim_{h\to 0} f(a+h)=na^{n-1}.$ ...(1)

Similarly we can find

$\text{LHL}=f(a-0)=\lim_{h\to 0} f(a-h)=na^{n-1}.$ ...(2)

Now, from (1) and (2) we find that

$$f(a+0)=f(a-0)=na^{n-1} \Rightarrow \lim_{x\to 0} f(x)=n\,a^{n-1}.$$

Q.6. Determine the values of *a, b, c* for which the function

$$f(x)=\begin{cases}\dfrac{\sin(a+1)x+\sin x}{x} & \text{for } x<0\\ c & \text{for } x=0\\ \dfrac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}} & \text{for } x>0\end{cases}$$

is continuous at $x=0$.

Ans. We have

$$\text{RHL}=f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}\frac{(h+bh^2)^{1/2}-h^{1/2}}{bh^{3/2}}=\lim_{h\to 0}\frac{(1+bh)^{1/2}-1}{bh}$$

$$=\lim_{h\to 0}\frac{[1+\frac{1}{2}bh+\ldots]-1}{bh}=\frac{1}{2}.$$

$$\text{LHL}=f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}\frac{\sin(a+1)(-h)+\sin(-h)}{(-h)}$$

$$=\lim_{h\to 0}\frac{\sin(a+1)h+\sin h}{h}=\lim_{h\to 0}\frac{2\sin\left(\frac{1}{2}a+1\right)h\cos\left(\frac{ah}{2}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\sin\{(a+2)/2\}h}{\{(a+2)/2\}h}(a+2)\cos(ah/2)=a+2.$$

Now, for continuity at $x=0$, we have $f(0+0)=f(0-0)=f(0)$

$\Rightarrow \qquad \frac{1}{2}=a+2=c \Rightarrow c=\frac{1}{2}, a=-\frac{3}{2}.$

Q.7. Show that the function *f* defined on R^+ as

$$f(x) = \sin\frac{1}{x}, \ \forall \ x > 0$$

is continuous, but not uniformly continuous on R^+.

Ans. Let $a \in R^+$.

We have $\text{LHL} = f(a-0) = \lim_{h\to 0} f(a-h) = \lim_{h\to 0} \sin\frac{1}{a-h} = \sin\frac{1}{a}$

$$\text{RHL} = f(a+0) = \lim_{h\to 0} f(a+h) = \lim_{h\to 0} \sin\frac{1}{a+h} = \sin\frac{1}{a}$$

$$f(a) = \sin\frac{1}{a} \quad \Rightarrow \quad f(a+0) = f(a) = f(a-0)$$

$\Rightarrow$ *f* is continuous at *a*.

Since, *a* is arbitrary point in R^+. Therefore, *f* is continuous on R^+.

Now, to show *f* is not uniformly continuous on R^+.

Let δ be any positive number. Take

$$x_1 = \frac{1}{n\pi}, x_2 = \frac{1}{n\pi + \pi/2} = \frac{2}{(2n+1)(\pi)} \text{ where } n \in Z^+$$

such that $x_1 - x_2 = \frac{1}{n\pi} - \frac{2}{(2n+1)\pi} < \delta.$

Now, $|x_1 - x_2| < \delta$ but $|f(x_1) - f(x_2)| = \left|\sin n\pi - \sin\frac{1}{2}(2n+1)\pi\right| = 1 > \varepsilon$

which shows that for this choice of ε, we can not find a $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \text{ for } |x_1 - x_2| < \delta \ \forall \, x_1, x_2 \in R^+.$$

Hence, *f* is not uniformly continuous on R^+.

Q.8. Write the Boundedness theorem and prove it.

Ans. Boundedness Theorem : If a function $f(x)$ is continuous in a closed interval $[a, b]$, then it is bounded in that interval.

Proof : By the above theorem, for a given $\varepsilon > 0$, we can sub-divide the interval $[a, b]$ into a finite number of sub-intervals say $[a = a_0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n = b]$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \quad \ldots(1)$$

for any two points x_1, x_2 in the same sub-interval. Let x be any point in the first sub-interval $[a, a_1]$. Then by (1), we have, $\forall \, x \in [a, a_1]$

$$|f(x) - f(a)| < \varepsilon \text{ i.e., } f(a) - \varepsilon < f(x) < f(a) + \varepsilon. \quad \ldots(2)$$

In particular, for $x = a_1, |f(a_1) - f(a)| < \varepsilon.$ ...(3)

Again $\forall \, x \in [a_1, a_2], |f(x) - f(a_1)| < \varepsilon.$ ...(4)

$\therefore$ $\forall \, x \in [a_1, a_2]$, we have

$$|f(x)-f(a)|=|f(x)-f(a_1)+f(a_1)-f(a)|$$
$$\le |f(x)-f(a_1)|+|f(a_1)-f(a)|$$
$$<\varepsilon+\varepsilon, \text{ from (3) and (4)} =2\varepsilon.$$

Thus $\forall\, x \in [a_1, a_2]$, we have $|f(x)-f(a)|<2\varepsilon$ *i.e.,* $f(a)-2\varepsilon<f(x)<f(a)+2\varepsilon$ ...(5)

From (2) and (5), we see that all the values of $f(x)$ in the first two sub-intervals lie between

$$f(a)-2\varepsilon \text{ and } f(a)+2\varepsilon.$$

Proceeding in the same way, we can show that $\forall\, x \in [a_{n-1}, a_n = b]$, we have

$$f(a)-n\varepsilon<f(x)<f(a)+n\varepsilon.$$

Hence all the values of $f(x)$ in the interval $[a,b]$ will be between $f(a)-n\varepsilon$ and $f(a)+n\varepsilon$.

Thus $f(x)$ is bounded in $[a, b]$.

Q.9. Write Darboux's theorem or Intermediate value theorem and prove it.

Ans. Theorem : If f is finitely differentiable in a closed interval $[a,b]$ and $f'(a)$, $f'(b)$ are of opposite sign, then there exists at least one point $c \in]a,b[$ such that $f'(c)=0$.

Proof : Let us suppose that $f'(a)>0$ and $f'(b)<0$, then there exists intervals $]a, a+h[$ and $]b-h, b[$, $h>0$, such that

$$f(x)>f(a), \forall\, x \in]a, a+h[\qquad \text{...(1)}$$
$$f(x)>f(b), \forall\, x \in]b-h, b[. \qquad \text{...(2)}$$

Now, since f is finitely differentiable, then it is continuous in $[a,b]$ and hence it is bounded on $[a,b]$ and attains its supremum and infimum at least once in $[a,b]$. [$\because$ A continuous function attains its supremum and infimum at least once in $[a,b]$].

Thus if M is the supremum of f in $[a,b]$, then there exists $c \in [a,b]$ such that $f(c)=M$. It is clear from (1) and (2) that the upper bound is not attained at the end points a and b so that $c \in]a,b[$.

Now, we shall prove $f'(c)=0$.

If $f'(c)>0$, then there exists an interval $]c, c+h]$, $h>0$, such that $f(x)>f(c)=M$, $\forall\, x \in]c, c+h[$, which is not possible, since M is the supremum of the function $f(x)$ in $[a,b]$.

If $f'(c)<0$, then there exists an interval $[c-h, c[$, $h>0$ such that $f(x)>f(c)=M$, $\forall\, x \in [c-h, c[$, which is not possible.

Hence, we conclude that $\mathbf{f'(c)=0}$.

Q.10. Show that the function $f: \mathbf{R} \to \mathbf{R}$ defined by

$$f(x)=\lim_{t\to\infty}\frac{(1+\sin \pi x)^t-1}{(1+\sin \pi x)^t+1}$$

is discontinuous at the points $x=0, 1, 2, \ldots, n, \ldots$

Ans. For $x=0,1,2,3,\ldots,n,\ldots$ we have $\sin \pi x=0$, so that at these values of x

$$f(x)=\lim_{n\to\infty}\frac{(1+0)^t-1}{(1+0)^t+1}=0.$$

Now if $2m<x<2m+1$ (m being an integer), then $\sin \pi x$ is positive. Hence for such values of x, we have

$$f(x)=\lim_{t\to\infty}\frac{1-\frac{1}{(1+\sin\pi x)^t}}{1+\frac{1}{(1+\sin\pi x)^t}}=\frac{1-\frac{1}{\infty}}{1+\frac{1}{\infty}}=1. \qquad \left[\because \frac{1}{\infty}=0\right]$$

Again if $2m+1<x<2m+2, \sin\pi x$ is negative and so

$$\lim_{t\to\infty}(1+\sin\pi x)^t=0.$$

Hence for such values of x, $f(x)=\frac{0-1}{0+1}=-1.$

From the values of $f(x)$ mentioned above, we observe that

(i) if x is an even integer, then

$$f(x)=0, f(x+0)=1 \text{ and } f(x-0)=-1$$

and (ii) if x is an odd integer, then

$$f(x)=0, f(x+0)=-1 \text{ and } f(x-0)=1.$$

Hence f has discontinuities of the first kind at $x=0,1,2,\ldots,n,\ldots$

Q.11. Prove that the function $f(x)=|x|$ is continuous at $x=0$, but not differentiable at $x=0$, where $|x|$ is the absolute value of x.

Ans. Firstly, we check the continuity of the function $f(x)$ at $x=0$.

We have $f(0)=|0|=0$

$$f(0+0)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}f(h)=\lim_{h\to 0}|h|=\lim_{h\to 0}h=0$$

$$f(0-0)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}f(-h)=\lim_{h\to 0}|-h|=\lim_{h\to 0}h=0.$$

$\therefore$ $f(0+0)=f(0)=f(0-0).$

Hence, $f(x)$ is continuous at $x=0$.

Now, we check the differentiability of the function $f(x)$ at $x=0$.

We have, $$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|-0}{h}=1$$

and $$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{|-h|-0}{-h}=\lim_{h\to 0}\frac{h}{-h}=-1$$

$\Rightarrow$ $Rf'(0)\neq Lf'(0).$

Hence, the function $f(x)$ is not differentiable at $x=0$.

Q.12. If $f(x)=\begin{cases} x^2 \sin\frac{1}{x}, & if \quad x \neq 0 \\ 0, & if \quad x=0 \end{cases}$

then, show that $f(x)$ is continuous and differentiable everywhere.

Ans. We have

$$f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}(0+h)^2 \sin\frac{1}{0+h}=\lim_{h\to 0} h^2 \sin\frac{1}{h}=0$$

$$f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}(0-h)^2 \sin\frac{1}{0-h}=-\lim_{h\to 0} h^2 \sin\frac{1}{h}=0$$

$$f(0)=0 \qquad \Rightarrow \qquad f(0+0)=f(0)=(0-0).$$

Hence, the function is continuous at $x=0$.

Now $$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{h^2 \sin\frac{1}{h}-0}{h}=\lim_{h\to 0} h\sin\frac{1}{h}=0$$

and $$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{(-h)^2 \sin\left(-\frac{1}{h}\right)-0}{-h}=\lim_{h\to 0} h\sin\frac{1}{h}=0$$

$\Rightarrow \qquad Rf'(0)=Lf'(0).$

Hence, $f(x)$ is differentiable at $x=0$.

Q.13. Show that the function $f:R\to R$ defined by

$$f(x)=\begin{cases} x\left[1+\frac{1}{3}\sin\log x^2\right] & if \quad x\neq 0 \\ 0 & if \quad x=0 \end{cases}$$

is continuous everywhere but not differentiable at origin.

Ans. Firstly, we check the continuity of $f(x)$ at $x=0$

$$f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}\left[(0+h)\left\{1+\frac{1}{3}\sin\log(0+h)^2\right\}\right]$$

$$=\lim_{h\to 0}\left[h+\left(\frac{h}{3}\right)\sin\log h^2\right]=0+0\times \text{a finite quantity}=0.$$

Similarly $f(0-0)=0$. Also, given that $f(0)=0$.

Hence, f is continuous at $x=0$.

Now we shell check the differentiability at $x=0$

$$Rf'(0)=\lim_{h\to 0}\frac{(0+h)\left\{1+\frac{1}{3}\sin\log(0+h)^2\right\}-0}{h}=\lim_{h\to 0}\left[1+\frac{1}{3}\sin\log h^2\right]$$

which does not exists, (since $\sin\log h^2$ oscillate between -1 and 1 as $h\to 0$)

$\neq 0.$

Similarly $Lf'(0)=$ does not exist.

Hence, $f(x)$ is not differentiable at origin.

Q.14. Write the chain rule of differentiability and prove it.

Ans. Theorem : Let f and g be functions such that the range of f is contained in the domain of g. If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then $g\,o\,f$ is differentiable at x_0, and

$$(g\,o\,f)'(x_0)=g'(f(x_0)).\,f'(x_0).$$

Proof : Let $y=f(x)$ and $y_0=f(x_0)$.

Since f is differentiable at x_0, we have

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0)\quad\text{or}\quad f(x)-f(x_0)=(x-x_0)[f'(x_0)+\lambda(x)]\quad ...(1)$$

where $\lambda(x)\to 0$ as $x\to x_0$.

Further since g is differentiable at y_0, we have

$$\lim_{y\to y_0}\frac{g(y)-g(y_0)}{y-y_0}=g'(y_0)\quad\text{or}\quad g(y)-g(y_0)=(y-y_0)[g'(y_0)+\mu(y)]\quad ...(2)$$

where $\mu(y)\to 0$ as $y\to y_0$.

Now $(g\,o\,f)(x)-(g\,o\,f)(x_0)=g(f(x))-g(f(x_0))=g(y)-g(y_0)$

$=(y-y_0)[g'(y_0)+\mu(y)]$, by (2)

$=[f(x)-f(x_0)][g'(y_0)+\mu(y)]$

$=(x-x_0)[f'(x_0)+\lambda(x)][g'(y_0)+\mu(y)]$, by (1).

Thus if $x\neq x_0$, then

$$\frac{(g\,o\,f)(x)-(g\,o\,f)(x_0)}{x-x_0}=[g'(y_0)+\mu(y)].[f'(x_0)+\lambda(x)].\quad ...(3)$$

Also f being differentiable at x_0, is continuous at x_0 and hence as $x\to x_0$, $f(x)\to f(x_0)$ *i.e.,* $y\to y_0$.

Consequently $\mu(y)\to 0$ as $x\to x_0$ and $\lambda(x)\to 0$ as $x\to x_0$.

Taking the limits as $x\to x_0$, we get from (3)

$$\lim_{x\to x_0}\frac{(g\,o\,f)(x)-(g\,o\,f)(x_0)}{x-x_0}=g'(y_0).\,f'(x_0).$$

Hence the function gof is differentiable at x_0 and $(gof)'(x_0)=g'(f(x_0))\,f'(x_0).$

SECTION-C LONG ANSWER TYPE QUESTIONS

Q.1. Let $I_n = \left]-\frac{1}{n}, 1+\frac{1}{n}\right[$ be an open interval for each $n \in N$. Prove that $\bigcap_{n=1}^{\infty} I_n$ is not a *nbd* of each of its points.

Ans. Since $n \in N$

$$\Rightarrow \qquad \frac{1}{n} \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \qquad -\frac{1}{n} \to 0 \text{ as } n \to \infty \text{ and } 1+\frac{1}{n} \to 1 \text{ as } n \to \infty.$$

Therefore, we can find that

$$0 \in \left]-\frac{1}{n}, 1+\frac{1}{n}\right[\ \forall\, n \in N. \quad \Rightarrow \quad 1 \in \left]-\frac{1}{n}, 1+\frac{1}{n}\right[\ \forall\, n \in N.$$

Also, each point lying between 0 and 1 is an element of the open interval

$$\left]-\frac{1}{n}, 1+\frac{1}{n}\right[\ \forall\, n \in N.$$

But $-\left(\frac{1}{n}\right)$, whatever be the value of n, is not an element of the open interval

$$\left]-\frac{1}{n}, 1+\frac{1}{n}\right[, n \in N$$

$\Rightarrow$ All numbers less than 0 are not in $\bigcap_{n=1}^{\infty} I_n$.

Similarly, we can show that all numbers greater than 1 are not $\bigcap_{n=1}^{\infty} I_n, n \in N$.

Now, $[0,1] = \bigcap_{n=1}^{\infty} \left\{\left]-\frac{1}{n}, 1+\frac{1}{n}\right[\right\} = \bigcap_{n=1}^{\infty} I_n$.

and [0, 1] being a closed interval is a nbd of each of the points of the interval [0, 1] except the end points 0 and 1.

Hence, $\bigcap_{n=1}^{\infty} I_n$ is not a nbd of each of its points.

Q.2. Find the limit points of the set S of rational numbers of the form $\left.\frac{n}{(n+1)} : n \in N\right\}$.

Ans. Here, we have $\qquad S = \left\{\frac{n}{n+1} : n \in N\right\}$

Also, $\dfrac{n}{n+1}=\dfrac{n+1-1}{n+1}=1-\dfrac{1}{n+1}$

Let $\varepsilon>0$ be arbitrary small positive number, then the nbd $]1-\varepsilon,1+\varepsilon[$ of the point 1 contains a point of *S*, other than 1, because by taking $n>\left[\dfrac{\varepsilon}{1-\varepsilon}\right]$, we have

$$\frac{n}{n+1}>\frac{\varepsilon/(1-\varepsilon)}{[\varepsilon/(1-\varepsilon)]+1} \quad\Rightarrow\quad \frac{n}{n+1}>\varepsilon$$

$\Rightarrow$ 1 is a limit point of the given set *A*.

Now, we check whether there is any other limit point of *S* other than 1.

Let us suppose $p\in A', p\neq 1$.

Now, these are following case :

Case (i) If $p > 1$

Then choose $\varepsilon>p-1$, then the nbd $]p-\varepsilon,p+\varepsilon[$ of *p* contains no point of *S*, other than *p*.

$\Rightarrow$ *p* is not the limit point of *S*.

Case (ii) If $p < 1$

$p\in A'$, then there exists a point of *S*, which is nearest to *p* and let p_r be this element of *S*, which is nearest to *p*. Choose a positive integer ε such that $\varepsilon<|p_r-p|$, then the nbd $]p-\varepsilon,p+\varepsilon[$ of the point *p* contains no point of *S* and so as before, we conclude that *p* is not the limit point of *S*.

Suppose that $p\in S$ and let $p=\dfrac{n}{n+1}$.

Then the point just before *p* is $\dfrac{n-1}{(n-1)+1}$ *i.e.,* $\dfrac{n-1}{n}$

and the point just after *p* is $\dfrac{n-1}{(n+1)+1}$ *i.e.,* $\dfrac{n+1}{n+2}$.

Now, we can find that

$$\frac{n+1}{n+2}-\frac{n}{n+1}=\frac{(n+1)^2-n(n+2)}{(n+1)(n+2)}=\frac{1}{(n+1)(n+2)}$$

and
$$\frac{n}{n+1}-\frac{n-1}{n}=\frac{n^2-(n-1)(n+1)}{n(n+1)}=\frac{1}{n(n+1)}.$$

Also $n+2<n$

$$\Rightarrow \qquad \frac{1}{n+2}<\frac{1}{n} \quad\Rightarrow\quad \frac{1}{(n+1)(n+2)}<\frac{1}{(n+1)n}$$

Hence, we have $\dfrac{n}{n+1}-\dfrac{n-1}{n}>\dfrac{n+1}{n+2}-\dfrac{n}{n+1}$

Let us choose a positive number $\varepsilon>0$ such that $\varepsilon<\left(\dfrac{n+1}{n+2}-\dfrac{n}{n+1}\right)$ then, the nbd $]p-\varepsilon,p+\varepsilon[$ of *p* contains no point of *S*, other than *p*.

$\Rightarrow$ p is not the limit point of S.

Hence, we find that no real number other than 1 is a limit point of A.

Q.3. Find the right hand and the left hand limits in the following cases and discuss the existence of the limit in each case :

(i) $\lim_{x\to 2} \dfrac{2x^2-8}{x-2}$; **(ii)** $\lim_{x\to 0} \dfrac{e^{1/x}-1}{e^{1/x}+1}$;

(iii) $\lim_{x\to 0} f(x)$ **where** $f(x)$ **is defined as**

$f(x)=x$, **when** $x>0$; $f(x)=0$, **when** $x=0$; $f(x)=-x$, **when** $x<0$.

Ans. (i) Let $f(x)=\dfrac{2x^2-8}{x-2}$

We have $$f(2+0)=\lim_{h\to 0} f(2+h)=\lim_{h\to 0}\frac{2(2+h)^2-8}{2+h-2}$$

$$=\lim_{h\to 0}\frac{2(4+4h+h^2)-8}{h}=\lim_{h\to 0}\frac{8h+2h^2}{h}$$

$$=\lim_{h\to 0}\frac{h(8+2h)}{h}=\lim_{h\to 0}(8+2h)=8.$$

Again $$f(2-0)=\lim_{h\to 0} f(2-h)=\lim_{h\to 0}\frac{2(2-h)^2-8}{2-h-2}$$

$$=\lim_{h\to 0}\frac{2(4-4h+h^2)-8}{-h}=\lim_{h\to 0}\frac{-8h+2h^2}{-h}$$

$$=\lim_{h\to 0}\frac{-h(8-2h)}{-h}=\lim_{h\to 0}(8-2h)=8.$$

Since $f(2+0)=f(2-0)=8$, therefore $\lim_{x\to 2}\dfrac{2x^2-8}{x-2}$ exists and is equal to 8.

(ii) Let $f(x)=\dfrac{e^{1/x}-1}{e^{1/x}+1}$.

Here the right hand limit, *i.e.*,

$$f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)=\lim_{h\to 0}\frac{e^{1/h}-1}{e^{1/h}+1}$$

$$=\lim_{h\to 0}\frac{e^{1/h}[1-(1/e^{1/h})]}{e^{1/h}[1+(1/e^{1/h})]}=1.$$

Again the left hand limit, *i.e.*,

$$f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)=\lim_{h\to 0}\frac{e^{-1/h}-1}{e^{-1/h}+1}$$

$$= \lim_{h\to 0} \frac{(1/e^{1/h})-1}{(1/e^{1/h})+1} = \frac{0-1}{0+1} = -1.$$

Since $f(0+0) \neq f(0-0)$, hence $\lim_{x\to 0} \frac{e^{1/x}-1}{e^{1/x}+1}$ does not exist.

(iii) We have the right hand limit *i.e.,*

$$f(0+0) = \lim_{h\to 0} f(0+h), \text{ where } h \text{ is + ive but sufficiently small}$$

$$= \lim_{h\to 0} f(h) \lim_{h\to 0} h, \qquad [\because h > 0 \text{ and } f(x) = x \text{ if } x > 0]$$

$$= 0.$$

Also, the left hand limit, *i.e.,*

$$f(0-0) = \lim_{h\to 0} f(0-h), \text{ where } h \text{ is + ive but sufficiently small}$$

$$= \lim_{h\to 0} f(-h) = \lim_{h\to 0} -(-h), \qquad [\because -h < 0 \text{ and } f(x) = -x \text{ if } x < 0]$$

$$= \lim_{h\to 0} h = 0.$$

Thus both the limits $f(0+0)$ and $f(0-0)$ exist and are equal to zero.

Hence $\lim_{x\to 0} f(x)$ exists and is equal to zero.

Q.4. Show that the function *f* defined as

$$f(x) = \begin{cases} 0, & \text{for } x = 0 \\ \frac{1}{2} - x, & \text{for } 0 < x < \frac{1}{2} \\ \frac{1}{2}, & \text{for } x = \frac{1}{2} \\ \frac{3}{2} - x, & \text{for } \frac{1}{2} < x < 1 \\ 1, & \text{for } x = 1 \end{cases}$$

has three points of discontinuity. Find such points. Also draw the graph of the function.

Ans. Here, we observe that the domain of the function $f(x)$ is closed interval $[0, 1]$ when $0 < x < \frac{1}{2}$, the function $f(x) = \frac{1}{2} - x$, which is being the polynomial is continuous at each points of its domain.

$\Rightarrow$ $f(x)$ is continuous at each point of the open interval $\left]0,\frac{1}{2}\right[$ when $\frac{1}{2}<x<1$,

$f(x)=\frac{3}{2}-x$, which is also a polynomial in x

$\Rightarrow$ $f(x)$ is continuous in the open interval $\left]\frac{1}{2},1\right[$.

Now, we check the continuity at $x=0,\frac{1}{2}$ and 1.

At $x=0$.

At $x=0$, $f(x)=0$

and RHL $=f(0+0)=\lim\limits_{h\to 0} f(0+h)=\lim\limits_{h\to 0} f(h)=\lim\limits_{h\to 0}\left(\frac{1}{2}-h\right)=\frac{1}{2}$

$\Rightarrow$ $f(0)\neq f(0+0)$

$\Rightarrow$ $f(x)$ is not continuous at $x=0$.

At $x=\frac{1}{2}$.

At $x=\frac{1}{2}$, $f(x)=\frac{1}{2}$

and RHL $=f\left(\frac{1}{2}-0\right)=\lim\limits_{h\to 0} f\left(\frac{1}{2}-h\right)=\lim\limits_{h\to 0}\left[\frac{1}{2}-\left(\frac{1}{2}-h\right)\right]=\lim\limits_{h\to 0} h=0$

$\Rightarrow$ $f\left(\frac{1}{2}\right)\neq f\left(\frac{1}{2}-0\right)$

$\Rightarrow$ $f(x)$ is not continuous at $x=\frac{1}{2}$.

At $x=1$.

At $x=1$, $f(x)=1$

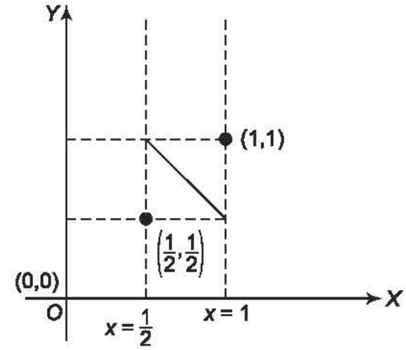
and LHL $=f(1-0)=\lim\limits_{h\to 0} f(1-h)=\lim\limits_{h\to 0}\left[\frac{3}{2}-(1-h)\right]=\lim\limits_{h\to 0}\left(\frac{1}{2}+h\right)=\frac{1}{2}$

$\Rightarrow$ $f(1)\neq f(1-0)$

$\Rightarrow$ $f(x)$ is not continuous at $x=1$.

Hence, the function $f(x)$ has three points of discontinuity given by $x=0,\frac{1}{2}$ and 1.

Graph of $f(x)$: The graph of the function consists of the point (0, 0), the segment of the line $y=\frac{1}{2}-x$ for $0<x<\frac{1}{2}$, the point $\left(\frac{1}{2},\frac{1}{2}\right)$, the segment of the line $y=\frac{3}{2}-x$ for $\frac{1}{2}<x<1$ and the point (1, 1). The graph of $f(x)$ is given as fig.



Q.5. Show that the function

$$f(x)=\begin{cases} x\left[\dfrac{1-e^{-2/x}}{1+e^{-2/x}}\right], & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$$

is continuous but not differentiable at $x=0$.

Ans. **Continuity of $f(x)$ at $x=0$.** We have

$$\text{RHL}=f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)=\lim_{h\to 0} h\left[\frac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}\right]$$

$$=\lim_{h\to 0} h\left[\frac{1-e^{-2/h}}{1+e^{-2/h}}\right]=0\times\frac{1-0}{1+0}=0\times 1=0$$

$$\text{LHL}=f(0-0)=\lim_{h\to 0} f(0-h)$$

$$=\lim_{h\to 0} f(-h)=\lim_{h\to 0} -h\left[\frac{e^{1/-h}-e^{-1/-h}}{e^{1/-h}+e^{-1/-h}}\right]=\lim_{h\to 0} -h\left[\frac{e^{1/-h}-e^{1/h}}{e^{-1/h}+e^{1/h}}\right]$$

$$=\lim_{h\to 0} -h\left[\frac{e^{-2/h}-1}{e^{-2/h}+1}\right]=0\times\frac{0-1}{0+1}=0 \quad\Rightarrow f(0+0)=f(0)=f(0-0).$$

Hence, $f(x)$ is continuous at $x=0$.

Differentiability of $f(x)$ at $x=0$.

We have

$$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{\left[h\dfrac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}-0\right]}{h}=\lim_{h\to 0}\frac{1-e^{-2/h}}{1+e^{-2/h}}=\frac{1-0}{1+0}=1$$

$$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{\left[(-h)\dfrac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}} - 0\right]}{-h} = \lim_{h\to 0} \frac{e^{-2/h}-1}{e^{2/h}+1} = \frac{0-1}{0+1} = -1$$

$\Rightarrow$ $Rf'(0) \neq Lf'(0)$.

Hence, the function $f(x)$ is not differentiable at $x = 0$.

Q.6. Test the continuity and differentiability in $-\infty < x < \infty$ of the following function :

$$f(x) = \begin{cases} 1 & \text{if } -\infty < x < 0 \\ 1+\sin x & \text{if } 0 \le x < \dfrac{\pi}{2} \\ 2+\left(x-\dfrac{\pi}{2}\right)^2 & \text{if } \dfrac{\pi}{2} \le x < \infty \end{cases}$$

Ans. Firstly, we check the continuity and differentiability at $x = 0$.

(1) Continuity of $f(x)$ at $x = 0$.

$$f(0) = 1 + \sin 0 = 1$$

$$f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} (1+\sin h) = 1$$

$$f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} 1 = 1$$

$\Rightarrow$ $f(0+0) = f(0) = f(0-0)$.

Hence, $f(x)$ is continuous at $x = 0$.

(2) Differentiability of $f(x)$ at $x = 0$.

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{(1+\sin h)-(1+\sin 0)}{h} = \lim_{h\to 0} \frac{\sin h}{h} = 1$$

and $$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{1-(1+\sin 0)}{-h} = \lim_{h\to 0} \frac{0}{-h} = \lim_{h\to 0} 0 = 0$$

$\Rightarrow$ $Rf'(0) \neq Lf'(0)$.

Hence, $f(x)$ is not differentiable at $x = 0$.

Now, we shall check the continuity and differentiability at $x = \frac{\pi}{2}$.

(3) Continuity of $f(x)$ at $x=\frac{\pi}{2}$.

We have $f\left(\frac{\pi}{2}\right)=2+\left(\frac{\pi}{2}-\frac{\pi}{2}\right)^2=2$

$$f\left(\frac{\pi}{2}+0\right)=\lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)=\lim_{h\to 0}\left[2+\left\{\left(\frac{1}{2}\pi+h\right)-\frac{1}{2}\pi\right\}^2\right]=\lim_{h\to 0}(2+h^2)=2;$$

$$f\left(\frac{\pi}{2}-0\right)=\lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)=\lim_{h\to 0}\left[1+\sin\left(\frac{\pi}{2}-h\right)\right]=\lim_{h\to 0}(1+\cos h)=1+1=2$$

$$\Rightarrow \quad f\left(\frac{\pi}{2}+0\right)=f\left(\frac{\pi}{2}\right)=f\left(\frac{\pi}{2}-0\right).$$

Hence, $f(x)$ is continuous at $x=\frac{\pi}{2}$.

(4) Differentiability of $f(x)$ at $x=\frac{\pi}{2}$.

$$Rf'\left(\frac{\pi}{2}\right)=\lim_{h\to 0}\frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\left[2+\left\{\frac{\pi}{2}+h-\frac{\pi}{2}\right\}^2\right]-\left[2+\left(\frac{\pi}{2}-\frac{\pi}{2}\right)^2\right]}{h}=\lim_{h\to 0}\frac{2+h^2-2}{h}=\lim_{h\to 0}h=0$$

$$Lf'\left(\frac{\pi}{2}\right)=\lim_{h\to 0}\frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h}=\lim_{h\to 0}\frac{1+\sin\left(\frac{\pi}{2}-h\right)-2}{-h}$$

$$=\lim_{h\to 0}\frac{-1+\cos h}{-h}=\lim_{h\to 0}\frac{1-\cos h}{h}=\lim_{h\to 0}\frac{2\sin^2(h/2)}{h}$$

$$=\lim_{h\to 0}\left[\frac{\sin h/2}{h/2}.\sin h/2\right]=\lim_{h\to 0}\left[\frac{\sin h/2}{h/2}\right]\lim_{h\to 0}[\sin h/2]=1\times 0=0$$

Since $Rf'\left(\frac{\pi}{2}\right)=Lf'\left(\frac{\pi}{2}\right)$.

$\Rightarrow$ $f(x)$ is differentiable at $x=\frac{\pi}{2}$.

Since, here, we checked the continuity and differentiability at $x=0$ and $\frac{\pi}{2}$.

It is obviously continuous and differentiable at all other points. ❑

UNIT II

SECTION-A VERY SHORT ANSWER TYPE QUESTIONS

Q.1. Discuss the applicability of Rolles theorem in the interval $[-1,1]$ to the function $f(x)=|x|$.

Ans. Here, we have $f(x)=|x|$

$$\begin{aligned} \Rightarrow \quad & \left.\begin{aligned} f(-1)&=1 \\ \text{and} \quad f(1)&=1 \end{aligned}\right\} \Rightarrow f(1)=f(-1). \end{aligned}$$

Now, the function $f(x)$ is continuous throughout the closed interval $[-1,1]$ but $f(x)$ is not differentiable at $x=0 \in]-1,1[$. Hence, Rolle's theorem is not satisfied (due to the second condition).

Q.2. If $a+b+c=0$, then show that the quadratic equation $3ax^2+2bc+c=0$ has at least one root in $]0, 1[$.

Ans. Let us define a function $f(x)$ such that

$$f(x)=ax^3+bx^2+cx+d.$$

Here we have $f(0)=d$ and $f(1)=a+b+c+d=d$. $\quad (\because a+b+c=0)$

Obviously, $f(x)$ is continuous and differentiable in $]0, 1[$ (being a polynomial).

Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem in $[0, 1]$. Hence, there is at least one value of x in the open interval $]0, 1[$ where $f'(x)=0$

i.e., $\quad 3ax^2+2bc+c=0$ has at least one root in $]0, 1[$.

Q.3. What is the expansion of $\log(1+x)$?

Ans. The expansion of $\log(1+x)$ is $x-\frac{x^2}{2}+\frac{x^3}{3}+\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots$

Q.4. Find the n^{th} differential coefficient of $\log(ax+x^2)$.

Ans. Let $y=\log(ax+x^2)=\log[x(a+x)]=\log x+\log(a+x)$

Differentiating n times, we get

$$\begin{aligned} y_n &= \frac{d^n}{dx^n}(\log x)+\frac{d^n}{dx^n}\log(a+x) \\ &= \frac{(-1)^{n-1}(n-1)!.1^n}{x^n}+\frac{(-1)^{n-1}(n-1)!1^n}{(x+a)^n} \\ &= (-1)^{n-1}.(n-1)!\left[\frac{1}{x^n}+\frac{1}{(x+a)^n}\right] \end{aligned}$$

Q.5. Define partial derivative or partial differential coefficient.

Ans. W know that the differential coefficient of $f(x)$ with respect to x is given by $\lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x)-f(x)}{\delta x}$, provided this limit exists, and it is denoted by $f'(x)$ or $\dfrac{d}{dx}[f(x)]$.

If $u = f(x, y)$ be a continuous function of two independent variables x and y, then the differential coefficient of u w.r.t. (regarding y as constant) is called the **partial derivative or partial differential co-efficient of u** w.r.t. x and is denoted by various symbols such as

$$\frac{\partial u}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), f_x,$$

Q.6. Define homogeneous function.

Ans. A function $f(x, y)$ is said to be homogeneous function of degree n, if the degree of each of its terms in x and y is equal to n. Thus $a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots + a_{n-1} x y^{n-1} + a_n y^n$ is homogeneous function in x and y of order n.

Q.7. If $u = f\left(\dfrac{y}{x}\right)$, show that $x\dfrac{\partial u}{\partial y} + y\dfrac{\partial u}{\partial y} = 0$.

Ans. We have $\quad u = f\left(\dfrac{y}{x}\right)$

Differentiating (1) partially w.r.t. x and y respectively, we get

$$\frac{\partial u}{\partial x} = f'\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right) \quad \Rightarrow \quad x\frac{\partial u}{\partial x} = -\frac{y}{x} f'\left(\frac{y}{x}\right)$$

and

$$\frac{\partial u}{\partial y} = f'\left(\frac{y}{x}\right)\cdot\frac{1}{x} \quad \Rightarrow \quad y\frac{\partial u}{\partial x} = \frac{y}{x} f'\left(\frac{y}{x}\right)$$

Adding (2) and (3), we get $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$

Q.8. If $x^y + y^x = a^b$. Find $\dfrac{dy}{dx}$.

Ans. Let $\quad f(x, y) = x^y + y^x - a^b \quad \Rightarrow \quad f(x, y) = 0.$

Therefore, $\quad \dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = -\dfrac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}.$

Q.9. If $x = r\cos\theta, y = r\sin\theta, z = f(x, y)$; prove that $\dfrac{\partial z}{\partial x} = \cos\theta\dfrac{\partial z}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial z}{\partial \theta}$.

Ans. We have $\quad x = r\cos\theta \quad$ and $\quad y = r\sin\theta$

$\Rightarrow \quad r = x^2 + y^2 \quad$ and $\quad \theta = \tan^{-1} y/x$

$\Rightarrow \quad \dfrac{\partial z}{\partial x} = \cos\theta \quad$ and $\quad \dfrac{\partial z}{\partial r} = \dfrac{-\sin\theta}{r}$

Putting these values in eqn. $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial z}{\partial \theta}\dfrac{\partial \theta}{\partial x} = \cos\theta\dfrac{\partial z}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial z}{\partial \theta}$

SECTION-B SHORT ANSWER TYPE QUESTIONS

Q.1. If $f(x)=(x-1)(x-2)(x-3)$ and $a=0, b=4$, find 'c' using Lagrange's mean value theorem.

Ans. We have

$$f(x)=(x-1)(x-2)(x-3)=x^3-6x^2+11x-6.$$

$\therefore$ $f(a)=f(0)=-6$ and $f(b)=f(4)=6.$

$\therefore$ $\dfrac{f(b)-f(a)}{b-a}=\dfrac{6-(-6)}{4-0}=\dfrac{12}{4}=3.$

Also $f'(x)=3x^2-12x+11$ gives $f'(c)=3c^2-12c+11.$

Putting these values in Lagrange's mean value theorem.

$$\frac{f(b)-f(a)}{b-a}=f'(c),(a<c<b), \text{ we get}$$

$$3=3c^2-12c+11 \quad \text{or} \quad 3c^2-12c+8=0$$

or $$c=\frac{12\pm\sqrt{(144-96)}}{6}=2\pm\frac{2\sqrt{3}}{3}.$$

As both of these values of c lie in the open interval $]0,4[$, hence both of these are the required values of c.

Q.2. Verify Cauchy's mean value for $f(x)=\sin x$ and $g(x)=\cos x$ in $\left[-\frac{\pi}{2},0\right]$.

Ans. It can be easily seen that $f(x)$ and $g(x)$ both are continuous on $\left[-\frac{\pi}{2},0\right]$ and differentiable on $\left]-\frac{\pi}{2},0\right[$.

Also, $g'(x)=-\sin x\neq 0$ for any point in the interval $\left]-\frac{\pi}{2},0\right[$.

Then, by Cauchy's mean value theorem, $\exists$ at least one $c\in\left]-\frac{\pi}{2},0\right[$ such that

$$\frac{f(0)-f\left(-\frac{\pi}{2}\right)}{g(0)-g\left(-\frac{\pi}{2}\right)}=\frac{f'(c)}{g'(c)}.$$

Putting all the values and after simplification, we have

$$\cot c=-1$$

$\Rightarrow$ $$c=-\pi/4.$$

Since $c=-\pi/4$ lies $]-\pi/2,0[$, hence, Cauchy mean value theorem is verified.

Q.3. State and prove Lagrange's mean value theorem.

Ans. Theorem : If a function $f(x)$ is

(i) continuous in a closed interval $[a,b]$, and

(ii) differentiable in the open interval $]a,b[$ *i.e.,* $a<x<b$, then there exists at least one value '*c*' of *x* lying in the open interval $]a,b[$ such that

$$\frac{f(b)-f(a)}{b-a}=f'(c).$$

Proof : Consider the function $\phi(x)$ defined by $\phi(x)=f(x)+Ax$, ...(1)

where *A* is a constant to be chosen such that $\phi(a)=\phi(b)$

i.e., $$f(a)+Aa=f(b)+Ab \quad \text{or} \quad A=-\frac{f(b)-f(a)}{b-a}. \qquad ...(2)$$

(i) Now the function *f* is given to be continuous on $[a,b]$ and the mapping $x\to Ax$ is continuous on $[a,b]$, therefore ϕ is continuous on $[a,b]$.

(ii) Also, since *f* is given to be differentiable on $]a,b[$ and the mapping $x\to Ax$ is differentiable on $]a,b[$, therefore, ϕ is differentiable on $]a,b[$.

(iii) By our choice of *A*, we have $\phi(a)=\phi(b)$.

From (i), (ii) and (iii), we find that ϕ satisfies all the conditions of Rolle's theorem on $[a,b]$. Hence there exists at least one point, say $x=c$, of the open interval $]a,b[$, such that $\phi'(c)=0$.

But $$\phi'(x)=f'(x)+A, \text{ from (1).}$$

$\therefore$ $$\phi'(c)=0 \Rightarrow f'(c)+A=0$$

or $$f'(c)=-A=\frac{f(b)-f(a)}{b-a}, \text{ from (2).}$$

This proves the theorem. It is usually known as the 'First Mean Value Theorem of Differential Calculus'.

Q.4. Verify Lagrange's mean value theorem for the function

$$f(x)=\sin x \quad in \left[0,\frac{\pi}{2}\right].$$

Ans. The function $f(x)=\sin x$ is continuous and differentiable on **R**. Hence it is continuous as well as differentiable in $[0,\pi/2]$. Then, by Lagrange's mean value theorem, there must exists at least one *c* in $]0,\pi/2[$ such that

$$\frac{f(\pi/2)-f(0)}{\pi/2-0}=f'(c). \qquad ...(1)$$

Here $$f(0)=0,\ f(\pi/2)=1$$

$$f'(x)=\cos x \quad \Rightarrow \quad f'(c)=\cos c.$$

Put all these values in (1), we have

$$\frac{1-0}{\pi/2}=\cos c \Rightarrow \cos c=\frac{2}{\pi} \Rightarrow c=\cos^{-1}\left(\frac{2}{\pi}\right).$$

Since, $0<2/\pi<1$, therefore the value of $c=\cos^{-1}\left(\frac{2}{\pi}\right)$ lies in $\left]0,\frac{\pi}{2}\right[$, so the required value of *c*. Hence, Lagrange's mean value theorem is verified.

Q.5. Use Cauchy's mean value theorem, to evaluate $\lim_{x\to 1}\left[\dfrac{\cos\frac{\pi x}{2}}{\log(1/x)}\right]$.

Ans. Let us suppose

$$f(x)=\cos\left(\frac{1}{2}\pi x\right),\ \ g(x)=\log x$$

$$a=x \ \text{ and } \ b=1.$$

Putting all these values in Cauchy's mean value theorem,

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)},\ a<c<b$$

we get
$$\frac{\cos\frac{\pi}{2}-\cos\frac{\pi x}{2}}{\log 1-\log x}=\frac{-\frac{1}{2}\pi\sin\left(\frac{\pi c}{2}\right)}{1/c}\ x<c<1.$$

Now, taking the limit as $x\to 1$, which give that $c\to 1$, we get

$$\lim_{x\to 1}\left\{\frac{0-\cos\left(\frac{1}{2}\pi x\right)}{\log(1/x)}\right\}=\lim_{c\to 1}\left\{\frac{-\frac{1}{2}\pi\sin\left(\frac{1}{2}\pi c\right)}{(1/c)}\right\}$$

or
$$\lim_{x\to 1}\left\{\frac{-\cos\left(\frac{1}{2}\pi x\right)}{\log(1/x)}\right\}=-\frac{1}{2}\pi \qquad \left(\because \sin\frac{1}{2}\pi c\to 1 \text{ as } c\to 1\right)$$

or
$$\lim_{x\to 1}\left\{\frac{\cos\left(\frac{1}{2}\pi x\right)}{\log(1/x)}\right\}=\frac{\pi}{2}.$$

Q.6. State and prove Cauchy's mean value theorem.

Ans. Cauchy's Mean Value Theorem : If two functions $f(x)$ and $g(x)$ are

(i) continuous in a closed interval $[a,b]$,

(ii) differential in the open interval $]a,b[$,

(iii) $g'(x)\neq 0$ for any point of the open interval $]a,b[$, then there exists at least one value c of x in the open interval $]a,b[$, such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)},a<c<b.$$

Proof : First we observe that as a consequence of condition (iii), $g(b)-g(a)\neq 0$. For if $g(b)-g(a)=0$ *i.e.,* $g(b)=g(a)$, then the function $g(x)$ satisfies all the conditions of Rolle's theorem in $[a,b]$ and consequently there is some x in $]a,b[$ for which $g'(x)=0$, thus contradicting the hypothesis that $g'(x)\neq 0$ for any point of $]a,b[$.

Now consider the function $F(x)$ defined on $[a,b]$, by setting

$$F(x)=f(x)+Ag(x), \quad ...(1)$$

where A is a constant to be chosen such that $F(a)=F(b)$

i.e., $$f(a)+Ag(a)=f(b)+Ag(b)$$

or $$-A=\frac{f(b)-f(a)}{g(b)-g(a)}. \quad ...(2)$$

Since $g(b)-g(a)\neq 0$, therefore A is a definite real number.

(i) Now f and g are continuous on $[a,b]$, therefore, F is also continuous on $[a,b]$.

(ii) Again, since f and g are differentiable on $]a,b[$, therefore F is also differentiable on $]a,b[$.

(iii) By our choice of A, $F(a)=F(b)$.

Thus the function $F(x)$ satisfies the conditions of Rolle's theorem in the interval $[a,b]$. Consequently there exists, at least one value, say c, of x in the open interval $]a,b[$ such that $F'(c)=0$.

But $F'(x)=f'(x)+Ag'(x)$, from (1).

$\therefore$ $F'(c)=0 \Rightarrow f'(c)+Ag'(c)=0$

or $$-A=\frac{f'(c)}{g'(c)}. \quad ...(3)$$

From (2) and (3), we get

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}.$$

Q.7. Find the second differential coefficient of $e^{3x}\sin 4x$.

Ans. Let $y=e^{3x}\sin 4x$.

Then $$\frac{dy}{dx}=3e^{3x}\sin 4x+4e^{3x}\cos 4x=e^{3x}(3\sin 4x+4\cos 4x).$$

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\{e^{3x}(3\sin 4x+4\cos 4x)\}$$

$$=3e^{3x}(3\sin 4x+4\cos 4x)+e^{3x}(12\cos 4x-16\sin 4x)$$

$$=e^{3x}(24\cos 4x-7\sin 4x).$$

Q.8. If $y=x\log\frac{x-1}{x+1}$, show that $y_n=(-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n}-\frac{x+n}{(x+1)^n}\right]$.

Ans. Let $y=x\log\frac{x-1}{x+1}$

$$\Rightarrow \quad y=x\log(x-1)-x\log(x+1) \quad ...(1)$$

Differentiating (1) w.r.t. x we get

$$y_1=\frac{x}{x-1}+\log(x-1)-\frac{x}{x+1}-\log(x+1)$$

$$=1+\frac{1}{x-1}+\log(x-1)-1+\frac{1}{x+1}-\log(x+1)$$

$$=\frac{1}{x-1}+\frac{1}{x+1}+\log(x-1)-\log(x+1) \qquad ...(2)$$

Differentiating both sides of (2) $(n-1)$ times w.r.t. x we get

$$y_n=\frac{(-1)^{n-1}(n-1)!}{(x-1)^n}+\frac{(-1)^{n-1}(n-1)!}{(x+1)^n}+\frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}}-\frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}}$$

$$=(-1)^{n-2}(n-2)!\left\{-\frac{(n-1)+n-1}{(x-1)^n}\right\}+(-1)^{n-2}(n-2)!\left\{\frac{-(n-1)-(x+1)}{(x+1)^n}\right\}$$

$$\Rightarrow \quad y_n=(-1)^{n-2}(n-2)!\left\{\frac{x-n}{(x-1)^n}-\frac{x+n}{(x+1)^n}\right\}$$

Q.9. If $y=\sin mx+\cos mx$, prove that $y_n=m^n[1+(-1)^n\sin 2mx]^{1/2}$.

Ans. We know that $\quad \frac{d^n}{dx^n}[\sin(ax+b)]=a^n\sin\left(n.\frac{\pi}{2}+ax+b\right)$

and $\quad \frac{d^n}{dx^n}[\cos(ax+b)]=a^n\cos\left(n.\frac{\pi}{2}+ax+b\right)$

Therefore, $y_n=\frac{d^n}{dx^n}(\sin mx)+\frac{d^n}{dx^n}(\cos mx)=m^n\sin\left(mx+n\frac{\pi}{2}\right)+m^n\cos\left(mx+n\frac{\pi}{2}\right)$

$$=m^n\left[\left\{\sin\left(mx+n\frac{\pi}{2}\right)+\cos\left(mx+n\frac{\pi}{2}\right)\right\}^2\right]^{1/2}$$

$$=m^n\left[1+2\sin\left(mx+n\frac{\pi}{2}\right).\cos\left(mx+n\frac{\pi}{2}\right)\right]^{1/2}$$

$$=m^n[1+\sin(2mx+n\pi)]^{1/2}=m^n[1\pm\sin 2mx]^{1/2}$$

$$=m^n[1+(-1)^n\sin 2mx]^{1/2}.$$

Q.10. Find the n^{th} differential coefficients of

(i) $\frac{1}{1-5x+6x^2}$ (ii) $\frac{x^2}{(x+2)(2x+3)}$

Ans. (i) Let $\quad y=\frac{1}{1-5x+6x^2}=\frac{1}{(3x-1)(2x-1)}$

$$=\frac{2}{2x-1}-\frac{3}{3x-1} \qquad \text{(By resolving into partial fractions)}$$

$$=2[2x-1]^{-1}-3[3x-1]^{-1}.$$

Differentiating, n times, we get

$$y_n = 2(-1)^n \, n!2^n \, (2x-1)^{-n-1} - 3\,(-1)^n \, n!3^n \, (3x-1)^{-n-1}$$

$$= (-1)^n \,. n![2^{n+1} \, (2x-1)^{-n-1} - 3^{n+1} \, (3x-1)^{-n-1}].$$

(ii) Let $y = \dfrac{x^2}{[x+2)\,(2x+3)]}$.

Since, the given fraction is not a proper one so, divide the Nr. by Dr., we observe that the quotient will be 1/2.

So let $\dfrac{x^2}{(x+2)\,(2x+3)} = \dfrac{1}{2} + \dfrac{A}{x+2} + \dfrac{B}{2x+3}$

which gives $A = -4, B = 9/2.$

Therefore, $y = \dfrac{1}{2} - \dfrac{4}{x+2} + \dfrac{9}{2\,(2x+3)} = \dfrac{1}{2} - 4\,(x+2)^{-1} + \dfrac{9}{2}(2x+3)^{-1}.$

Differentiating n times, we get

$$y_n = -4\,(-1)^n \, n!(x+2)^{-n-1} + \frac{9}{2}(-1)^n \,. n!2^n \, (2x+3)^{-n-1}$$

$$= (-1)^n \, n!\left[\frac{9.2^{n-1}}{(2x+3)^{n+1}} - \frac{4}{(x+2)^{n+1}}\right].$$

Q.11. Find the n^{th} differential coefficient of $x^3 \cos x$.

Ans. Let $u = \cos x$ and $v = x^3$

$\Rightarrow \quad u_n = \cos\left(x + \dfrac{n\pi}{2}\right), \; v_1 = 3x^2$

$u_{n-1} = \cos\left[x + \left((n-1)\dfrac{\pi}{2}\right)\right], \qquad v_2 = 6x$

$u_{n-2} = \cos\left(x + (n-2)\dfrac{\pi}{2}\right), \qquad v_3 = 6$

$u_{n-3} = \cos\left(x + (n-3)\dfrac{\pi}{2}\right), \qquad v_4 = 0.$

Now, by Leibnitz theorem, we have

$$\frac{d^n}{dx^n}(uv) = u_n \,. v + {}^nC_1 \, u_{n-1} \,. v_1 + {}^nC_2 \, u_{n-2} \,. v_2 + {}^nC_3 \, y_{n-3} \,. v_3$$

$$\Rightarrow \quad \frac{d^n}{dx^n}(x^3 \cos x) = \cos\left(x + \frac{n\pi}{2}\right)x^3 + {}^nC_1 \cos\left(x + (n-1)\frac{\pi}{2}\right)3x^2$$

$$+ {}^nC_2 \cos\left(x + \frac{(n-2)\pi}{2}\right)6x + {}^nC_3 \cos\left(x + \frac{(n-3)\pi}{2}\right)6$$

$$=x^3 \cos\left(x+\frac{n\pi}{2}\right)-3x^2 .n\sin\left(x+\frac{n\pi}{2}\right)-3n(n-1)x\cos\left(x+\frac{n\pi}{2}\right)$$

$$-n(n-1)(n-2)\sin\left(x+\frac{n\pi}{2}\right)$$

$$=[x^3-3n(n-1)x]\cos\left(x+\frac{n\pi}{2}\right)+[3x^2n-n(n-1)(n-2)]\sin\left(x+\frac{n\pi}{2}\right).$$

Q.12. If $y=a\cos(\log x)+b\sin(\log x)$, show that

$$x^2y_2+xy_1+y=0,$$

and $$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0.$$

Ans. We have $y=a\cos(\log x)+b\sin(\log x)$. ...(1)

$\therefore$ $y_1=-(a/x)\sin(\log x)+(b/x)\cos(\log x)$

or $xy_1=-a\sin(\log x)+b\cos(\log x)$. ...(2)

Differentiating (2) with respect to x, we have

$$xy_2+y_1=-(a/x)\cos(\log x)-(b/x)\sin(\log x)$$

or $x^2y_2+x_1=-y$ [From (1)]

or $x^2y_2+xy_1+y=0$. ...(3)

Differentiating (3) n times by Leibnitz's theorem, we have

$$D^n(x^2y_2)+D^n(xy_1)+D^n(y)=0$$

or $(D^ny_2)\cdot x^2+{}^nC_1(D^{n-1}y_2)\cdot(Dx^2)+{}^nC_2(D^{n-2}y_2)\cdot(D^2x^2)$

$$+(D^ny_1)\cdot x+{}^nC_1(D^{n-1}y_1)\cdot(Dx)+D^ny=0$$

or $\left[y_{n+2}x^2+n\cdot y_{n+1}\cdot 2x+\frac{n(n-1)}{2!}y_n\cdot 2\right]+[y_{n+1}\cdot x+ny_n\cdot 1]+y_n=0$

or $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$.

Q.13. If $y=e^{a\sin^{-1}x}$, show that

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0.$$

Ans. We have $y=e^{a\sin^{-1}x}\Rightarrow y_1=e^{a\sin^{-1}x}.\frac{a}{\sqrt{1-x^2}}$

$\Rightarrow$ $y_1\sqrt{1-x^2}=ae^{a\sin^{-1}x}=ay\Rightarrow y_1^2(1-x^2)=a^2y^2$. ...(1)

Now, differentiating (1) with respect to x, we get

$$2y_1y_2(1-x^2)+y_1^2(-2x)=2a^2yy_1$$

$\Rightarrow$ $2y_1[y_2(1-x^2)-xy_1-a^2y]=0$ [$\because 2y_1\neq 0$]

$\Rightarrow$ $[y_2(1-x^2)-xy_1-a^2y]=0$. ...(2)

Using Leibnitz's theorem, differentiating (2), n times, we get

$$D^n\,[y_2\,(1-x^2)]-D^n\,(y_1x)-a^2D^n\;y=0$$

$$\Rightarrow\quad \left[y_{n+2}\,(1-x^2)+ny_{n+1}\,(-2x)+\frac{n\,(n-1)}{2!}\,y_n\,(-2)\right]-[y_{n+1}\,x+ny_n]-a^2\,y_n=0$$

$$\Rightarrow\quad (1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}-(n^2+a^2)\,y_n=0.$$

Q.14. If $\cos^{-1}\left(\frac{y}{b}\right)=\log\left(\frac{x}{n}\right)^n$. Prove that $x^2\,y_{n+2}+(2n+1)\,xy_{n+1}+2n^2\,y_n=0.$

Ans. We have $\quad \cos^{-1}\left(\frac{y}{b}\right)=\log\left(\frac{x}{n}\right)^n=n\log\frac{x}{n}=n\,(\log x-\log n).$

Now, differentiating with respect to x, we get

$$-\frac{1}{\sqrt{\left[1-\frac{y^2}{b^2}\right]}}\,\frac{y_1}{b}=\frac{n}{x}\quad\text{or}\quad-\frac{y_1}{\sqrt{b^2-y^2}}=\frac{n}{x}\quad\text{or}\quad y_1{}^2x^2=n^2\,(b^2-y^2).$$

Again, differentiating, with respect to x, we get

$$2y_1\,y_2x^2+2xy_1{}^2=-2n^2\,yy_1$$

or $\qquad y_2x^2+y_1x+n^2\,y=0. \qquad (\because\; 2y_1\neq 0)$

Using Leibnitz's theorem, differentiating n times, we get

$$y_{n+2}x^2+{}^nC_1\;y_{n+1}\,(2x)+{}^nC_2\;y_2\,(2)+y_{n+1}x+{}^nC_1\;y_n+n^2\,y_n=0$$

$$\Rightarrow\quad x^2\;y_{n+2}+(2n+1)\,xy_{n+1}+2n^2\,y_n=0.$$

Q.15. State and prove Maclaurin's theorem.

Ans. Maclaurin's Theorem : Let $f(x)$ be a function of x which possesses continuous derivatives of all orders in the interval $[0, x]$ and can be expanded as an infinite series in x, then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}\,f''(0)+\ldots+\frac{x^n}{n!}\,f^n(0)+\ldots$$

Proof : Let us define

$$f(x)=A_0+A_1x+A_2x^2+A_3x^3+\ldots \qquad \ldots(1)$$

Let the expression (1) be differentiable term by term any number of times. Then by successive differentiation, we have

$$f'(x)=A_1+2A_2x+3A_3x^2+4A_4x^3+\ldots$$

$$f''(x)=2.1.A_2+3.2.A_3x+4.3\,A_4x^2+\ldots$$

$$f'''(x)=3.2.1.A_3+4.3.2\,A_4x+\ldots$$

....

Putting $x=0$, we get

$$f(0)=A_0,\,f'(0)=A_1,\,f''(0)=2!\,A_2,\,f'''(0)=3!\,A_3\,\ldots$$

$$\Rightarrow \qquad A_0 = f(0), A_1 = f'(0), A_2 = \frac{f''(0)}{2!}, A_3 = \frac{f'''(0)}{3!}\ldots$$

Substitute all these values in (1), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^n(0) + \ldots$$

This is Maclaurin's Theorem.

Q.16. Prove by Maclaurin's theorem, that $e^{\sin x} = 1 + x + \dfrac{x^2}{1.2} - \dfrac{3.x^4}{1.2.3.4} + \ldots$

Ans. Let $f(x) = e^{\sin x} \Rightarrow f(0) = e^0 = 1$

$$f'(x) = e^{\sin x}.\cos x \Rightarrow f'(0) = e^0 \cos 0 = 1$$

$$f'' = e^{\sin x}(-\sin x) + \cos x\, e^{\sin x} \cos x$$

$$= e^{\sin x}[\cos^2 x - \sin x] \Rightarrow f''(0) = e^0[1-0] = 1$$

$$f'''(x) = e^{\sin x}[2\cos x(-\sin x) - \cos x] + e^{\sin x}\cos x.[\cos^2 x - \sin x]$$

$$= e^{\sin x}\cos x[-2\sin x - 1 + \cos^2 x - \sin x]$$

$$= -e^{\sin x}\cos x[3\sin x + \sin^2 x] \Rightarrow f'''(0) = 0$$

$$f^{iv}(x) = -e^{\sin x}\cos x[3\cos x + 2\sin x\cos x] + e^{\sin x}\sin x[3\sin x + \sin^2 x]$$
$$-[3\sin x + \sin^2 x]\cos x\, e^{\sin x}.\cos x$$

$$\Rightarrow \qquad f^{iv}(0) = -3$$

Putting all these values in Maclaurin's theorem, given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \ldots$$

we get, $$e^{\sin x} = 1 + x + \frac{x^2}{1.2} - \frac{3x^4}{1.2.3.4} + \ldots$$

Q.17. Expand $\sin x$ in powers of $(x - \frac{1}{2}\pi)$ by using Taylor's series.

Ans. Let $f(x) = \sin x$. We want to expand $f(x)$ in powers of $x - \frac{1}{2}\pi$.

We can write $f(x) = f[\frac{1}{2}\pi + (x - \frac{1}{2}\pi)]$.

Now expanding $f[\frac{1}{2}\pi + (x - \frac{1}{2}\pi)]$ by Taylor's theorem in powers of $(x - \frac{1}{2}\pi)$, we get

$$f(x) = f[\frac{1}{2}\pi + (x - \frac{1}{2}\pi)] = f(\pi/2) + (x - \frac{1}{2}\pi) f'(\pi/2)$$
$$+ \frac{1}{2!}(x - \frac{1}{2}\pi)^2 f''(\pi/2) + \frac{1}{3!}(x - \frac{1}{2}\pi)^3 f'''(\pi/2) + \ldots \qquad \ldots(1)$$

Now $f(x)=\sin x$. Therefore $f(\pi/2)=\sin(\pi/2)=1$,

$f'(x)=\cos x$ giving $f'(\pi/2)=\cos(\pi/2)=0$,

$f''(x)=-\sin x$ so that $f''(\pi/2)=-\sin(\pi/2)=-1$,

$f'''(x)=-\cos x$ so that $f'''(\pi/2)=-\cos(\pi/2)=0$,

$f^{(iv)}(x)=\sin x$ so that $f^{(iv)}(\pi/2)=\sin(\pi/2)=1$, etc.

Substituting these values in (1), we get

$$\sin x=1+(x-\frac{1}{2}\pi)\cdot 0+\frac{1}{2!}(x-\frac{1}{2}\pi)^2\cdot(-1)+\frac{1}{3!}(x-\frac{1}{2}\pi)^3\cdot 0+\frac{1}{4!}(x-\frac{1}{2}\pi)^4\cdot 1+\ldots$$

$$=1-\frac{1}{2!}(x-\frac{1}{2}\pi)^2+\frac{1}{4!}(x-\frac{1}{2}\pi)^4\cdot 1-\ldots$$

Q.18. Expand by Maclaurin's Theorem $\frac{e^x}{1+e^x}$ as far as the term x^3.

Ans. Let $y=\frac{e^x}{1+e^x}=\frac{1+e^x-1}{1+e^x}=1-\frac{1}{1+e^x}$

Then $(y)_0=\frac{e^0}{1+e^0}=\frac{1}{2}$

$$y_1=0+\frac{e^x}{(1+e^x)^2}=\frac{e^x}{(1+e^x)}\cdot\frac{1}{(1+e^x)}=y(1-y)=y-y^2$$

$\Rightarrow$ $(y_1)_0=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$

Now $y_2=y_1-2yy_1 \Rightarrow (y_2)_0=\frac{1}{4}-2.\frac{1}{2}.\frac{1}{4}=0$

$y_3=y_2-2y_1^2-2yy_2$

$\Rightarrow$ $(y_3)_0=0-2\left(\frac{1}{4}\right)^2-0=-\frac{1}{8}$ and so on

Putting all these values in Maclaurin's series, we get

$$\frac{e^x}{1+e^x}=\frac{1}{2}+x.\frac{1}{4}+\frac{x^2}{21}.0+\frac{x^3}{3!}\left(-\frac{1}{8}\right)+\ldots=\frac{1}{2}+\frac{x}{4}-\frac{1}{48}x^3+\ldots$$

Q.19. If $f(x)=x^3+8x^2+15x-24$, calculate the value of $f\left(\frac{11}{10}\right)$ by Taylor's theorem.

Ans. By Taylor's Theorem, we have

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots \quad \ldots(1)$$

We want to find $f\left(\frac{11}{10}\right)$, *i.e.,* $f\left(1+\frac{1}{10}\right)$

Put $x=1$ and $h=\frac{1}{10}$ in (1), we get

$$f\left(\frac{11}{10}\right)=f\left(1+\frac{1}{10}\right)=f(1)+\frac{1}{10}f'(1)+\frac{1}{10^2}.\frac{1}{2!}f''(1)+\frac{1}{3!}\frac{1}{(10)^3}f'''(1)+\ldots \quad ...(2)$$

Now

$$f(x)=x^3+8x^2+15x-24 \quad \Rightarrow \quad f(1)=0$$

$$f'(x)=3x^2+16x+15 \quad \Rightarrow \quad f'(1)=34$$

$$f''(x)=6x+16 \quad \Rightarrow \quad f''(1)=22$$

$$f'''(x)=6 \quad \Rightarrow \quad f'''(1)=6$$

$$f^{iv}(x)=0 \quad \Rightarrow \quad f^{iv}(0)=0$$

Put all these values in (2), we get

$$f\left(1+\frac{1}{10}\right)=0+\frac{1}{10}.34+\frac{11}{100}+\frac{1}{1000}=3\cdot 4+0\cdot 11+0\cdot 001=3\cdot 511.$$

Q.20. If $u=\log(x^3+y^3+z^3-3xyz)$, show that

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z} \text{ and } \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\frac{-9}{(x+y+z)^2}.$$

Ans. We have $u=\log(x^3+y^3+z^3-3xyz)$.

$$\therefore \frac{\partial u}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}, \frac{\partial u}{\partial y}=\frac{3y^2-3zx}{x^3+y^3+z^3-3xyz} \text{ and } \frac{\partial u}{\partial y}=\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}.$$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{2(x^2+y^2+z^2-yz-zx-xy)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-yz-zx-xy)}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)}=\frac{3}{x+y+z} \quad ...(1)$$

Now, $$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)$$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right), \quad \text{[From (1)]}$$

$$=3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$=3\left[\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}\right]=\frac{-9}{(x+y+z)^2}.$$

Q.21. If $\theta = t^n e^{-r^2/4t}$, what value of n will make $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$?

Ans. We have $\dfrac{\partial \theta}{\partial r} = t^n . e^{-r^2/4t} . \left(-\dfrac{2r}{4t}\right) = -\dfrac{r}{2} t^{n-1} e^{-r^2/4t}$.

$$\therefore \quad r^2 \frac{\partial \theta}{\partial r} = -\frac{1}{2} r^3 t^{n-1} e^{-r^2/4t}.$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{3r^2}{2} t^{n-1} e^{-r^2/4t} - \frac{1}{2} r^3 t^{n-1} e^{-r^2/4t} . \left(-\frac{2r}{4t}\right)$$

$$= -\frac{3}{2} r^2 t^{n-1} e^{-r^2/4t} + \frac{1}{4} r^4 t^{n-2} e^{-r^2/4t}.$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{3}{2} t^{n-1} e^{-r^2/4t} + \frac{1}{4} r^2 t^{n-2} e^{-r^2/4t}.$$

Also $\dfrac{\partial \theta}{\partial t} = nt^{n-1} e^{-r^2/4t} + t^n e^{-r^2/4t} . \dfrac{r^2}{4t^2} = nt^{n-1} e^{-r^2/4t} + \dfrac{1}{4} r^2 t^{n-2} e^{-r^2/4t}$.

Now $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$$

$$\Rightarrow \quad -\frac{3}{2} t^{n-1} e^{-r^2/4t} + \frac{1}{4} r^2 t^{n-2} e^{-r^2/4t} = n t^{n-1} e^{-r^2/4t} + \frac{1}{4} r^2 t^{n-2} e^{-r^2/4t}$$

$\Rightarrow \quad -\dfrac{3}{2} t^{n-1} e^{-r^2/4t} = n t^{n-1} e^{-r^2/4t}$, for all possible values of r and t

$$\Rightarrow \quad n = -\frac{3}{2}.$$

Q.22. State and prove Euler's theorem on homogeneous functions.

Ans. Euler's theorem on homogeneous functions : If u is a homogeneous function of x and y of degree n, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$.

Proof : Since u is a homogeneous function of x and y of degree n, therefore u may be put in the form $u = x^n f(y/x)$. ...(1)

Differentiating (1) partially w.r.t. 'x', we have

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[x^n f(y/x)] = [f(y/x)] nx^{n-1} + x^n [f'(y/x)](-y/x^2).$$

$$\therefore \quad x\frac{\partial u}{\partial x} = nx^n f(y/x) - x^{n-1} y . f'(y/x). \qquad ...(2)$$

Again differentiating (1) partially w.r.t. 'y', we have

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[x^n f(y/x)] = x^n [f'(y/x)] . \frac{1}{x} = x^{n-1} f'(y/x).$$

$$\therefore \quad y\frac{\partial u}{\partial y} = y . x^{n-1} f'(y/x). \qquad ...(3)$$

Adding (2) and (3), we have

$$y\frac{\partial u}{\partial y}+y\frac{\partial u}{\partial y}=nx^n f(y/x)=nu. \qquad \text{[From (1)]}$$

Hence
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu.$$

This proves the theorem.

SECTION-C LONG ANSWER TYPE QUESTIONS

Q.1. State and prove Rolle's Theorem.

Ans. If a function f defined on $[a,b]$ is such that it is

(i) continuous in the closed interval $[a,b]$,

(ii) differentiable in the open interval $]a,b[$ and

(iii) $f(a)=f(b)$,

then there exists at least one value of x, say c, $(a<c<b)$ such that

$$f'(c)=0.$$

Proof : Since, the function $f(x)$ is continuous on $[a,b]$

$\Rightarrow$ $f(x)$ is bounded [$\because$ Every continuous function is bounded]

$\Rightarrow$ $f(x)$ attains its bounds [$\because$ A function, which is continuous on a closed bounded interval $[a,b]$, then it attains its bound on $[a,b]$]

Let M and m are the supremum and infimum of $f(x)$ respectively.

Now there are two possibilities

(i) $M=m$ **(ii) $M\neq m$.**

(i) If $M=m$, then obviously $f(x)$ is a constant function, and therefore its derivative is zero, *i.e.*

$$f'(x)=0\ \forall\ x\in\ ]a,b[.$$

(ii) If $M\neq m$, then at least one of the numbers M and m must be different from the equal values $f(a)$ and $f(b)$.

Let us assume $M\neq f(a)$.

Now, since, every continuous function on a closed interval attains its supremum, therefore, there exists a real number c in $[a,b]$ such that $f(c)=M$.

Also since $f(a)\neq m\neq f(b)$.

Therefore $c\neq a$ and $c\neq b$, this implies that $c\in\ ]a,b[$.

Now, $f(c)$ is the supremum of f on $[a,b]$

$\therefore$
$$f(x)\leq f(c)\ \forall\ x\in[a,b] \qquad ...(1)$$

(By the definition of supremum)

In particular, $f(c-h)\leq f(c)\quad h>0.$

$\Rightarrow$
$$\frac{f(c-h)-f(c)}{-h}\geq 0 \qquad ...(2)$$

Since $f'(x)$ exists at each point of $]a,b[$, and hence, $f'(c)$ exists.

Hence, from (2) $Lf'(c) \geq 0.$ ...(3)

Similarly from (1) $f(c+h) \leq f(c),\ h > 0.$

Then by the same arguments

$$Rf'(c) \leq 0. \quad \text{...(4)}$$

Since $f(x)$ is differentiable in $]a,b[\ \Rightarrow\ f'(c)$ exists

$\Rightarrow$ $Lf'(c) = f'(c) = Rf'(c).$ ...(5)

Now from (3), (4) and (5) $f'(c) = 0.$

Similarly we can consider the case

$$M = f(a) \neq m.$$

Q.2. Discuss the applicability of Rolle's theorem to $f(x) = \log\left[\dfrac{x^2+ab}{(a+b)x}\right]$, in the interval $[a, b], 0 < a < b$.

Ans. Here $f(a) = \log\left[\dfrac{a^2+ab}{(a+b)a}\right] = \log 1 = 0,$ and $f(b) = \log\left[\dfrac{b^2+ab}{(a+b)b}\right] = \log 1 = 0.$

Thus $f(a) = f(b) = 0.$

Also $Rf'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{1}{h}\left[\log\left\{\dfrac{(x+h)^2+ab}{(a+b)(x+h)}\right\} - \log\left\{\dfrac{x^2+ab}{(a+b)x}\right\}\right]$

$$= \lim_{h\to 0} \frac{1}{h}\left[\log \frac{(x^2+2xh+h^2+ab)(a+b)x}{(a+b)(x+h)(x^2+ab)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\log\left\{\frac{(x^2+2xh+h^2+ab)}{x^2+ab} \times \frac{x}{x+h}\right\}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\log\left\{1+\frac{2xh+h^2}{x^2+ab}\right\} - \log\left\{1+\frac{h}{x}\right\}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{2xh+h^2}{x^2+ab} - \frac{h}{x} + \ldots\right], \quad \text{...(1)} \quad \left[\because \log(1+y) = y - \frac{1}{2}y^2 + \ldots\right]$$

$$= \frac{2x}{x^2+ab} - \frac{1}{x}.$$

Again $Lf'(x) = \lim\limits_{h\to 0}\left[\dfrac{f(x-h)-f(x)}{-h}\right] = \lim\limits_{h\to 0} \dfrac{1}{(-h)}\left[\dfrac{-2hx+h^2}{x^2+ab} - \dfrac{(-h)}{x} + \ldots\right],$

replacing h by $-h$ in (1)

$$= \frac{2x}{x^2+ab} - \frac{1}{x}.$$

Since $Rf'(x) = Lf'(x)$, $f(x)$ is differentiable for all values of x in $[a,b]$. This implies that $f(x)$ is also continuous for all values of x in $[a,b]$. Thus all the three conditions of Rolle's theorem are satisfied. Hence $f'(x) = 0$ for at least one value of x in the open interval $]a,b[$.

Now $f'(x)=0 \Rightarrow \frac{2x}{x^2+ab}-\frac{1}{x}=0$ or $2x^2-(x^2+ab)=0$ or $x^2=ab$ or $x=\sqrt{(ab)}$,

which being the geometric mean of a and b lies in the open interval $]a,b[$. Hence the Rolle's theorem is verified.

Q.3. State and prove Taylor's theorem.

Ans. Let $f(x)$ be a single valued function defined on $[a,a+h]$ such that

(i) All the derivative of $f(x)$ upto $(n-1)^{th}$ are continuous in $[a,a+h]$

and (ii) $f^n(x)$ exists in $a<x<a+h$

even there exists a real number $\theta, 0<\theta<1$, such that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h)$$

where p is a given positive integer.

Proof : Since f^{n-1} exists, all the derivative $f', f''\ldots f^{n-1}$ exist and continuous on $[a,a+h]$, consider a function ϕ defined on $[a,a+h]$ such that

$$\phi(x)=f(x)+(a+h-x)f'(x)+\frac{(a+h-x)^2}{2!}f''(x)+\ldots$$
$$+\frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x)+A(a+h-x)^p \quad \ldots(1)$$

where A is a constant to be determined such that

$$\phi(a+h)=\phi(a)$$

Now $\phi(a)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+Ah^p$

and $\phi(a+h)=f(a+h)$

$$\therefore \quad f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+Ah^p. \quad \ldots(2)$$

Now

(i) $f, f', f''\ldots f^{n-1}$ being all continuous on $(a,a+h)$

$\Rightarrow$ The function ϕ is continuous on $[a,a+h]$,

(ii) Similarly the function ϕ is differentiable on $]a,a+h[$, and

(iii) $\phi(a+h)=\phi(a)$.

Thus, the function ϕ satisfied all the conditions of Rolle's theorem and hence $\exists$ a real number θ $(0<\theta<1)$ such that

$$\phi'(a+\theta h)=0.$$

Here $\phi'(x)=f'(x)+[-f'(x)+(a+h-x)f''(x)]$

$$+\frac{1}{2!}[-2(a+h-x)f''(x)+(a+h-x)^2 f'''(x)]+\ldots$$
$$+\frac{1}{(n-1)!}[-(n-1)(a+h-x)^{n-2}f^{n-1}(x)$$
$$+(a+h-x)^{n-1}f^n(x)]-Ap(a+h-x)]^{p-1}$$

$$= \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - Ap(a+h-x)^{p-1} \quad \text{[Other terms canceled in pairs]}$$

$$\therefore \qquad 0 = \phi'(a+\theta h) = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) - Aph^{p-1}(1-\theta)^{p-1}$$

$$\Rightarrow \qquad A = \frac{h^{n-p}(1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta h), h \neq 0, \theta \neq 1.$$

Now, putting the values of A in (2), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n (1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta h).$$

Q.4. Find the n^{th} differential coefficient of $\tan^{-1}\frac{x}{a}$.

Ans. We have

$$y = \tan^{-1}\frac{x}{a} \quad \Rightarrow \quad y_1 = \frac{a}{x^2+a^2} = \frac{a}{(x+ia)(x-ia)}.$$

Let us suppose

$$\frac{a}{(x+ia)(x-ia)} = \frac{A}{(x+ia)} + \frac{B}{(x-ia)} \qquad \text{(Using partial fractions)}$$

$$\Rightarrow \qquad a = A(x-ia) + B(x+ia).$$

To find the value of A, put $x = -ia$

we get $$A = -\frac{1}{2i}$$

and for B, put $x = ia$, which gives $B = \frac{1}{2i}$ therefore, we have

$$y_1 = \frac{1}{2i}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right] = \frac{1}{2i}[(x-ia)^{-1} - (x+ia)^{-1}].$$

Differentiating $(n-1)$ times, we get

$$y_n = \frac{1}{2!}[(-1)^{n-1}(n-1)!(x-ia)^{-n} - (-1)^{n-1}(n-1)!(x+ia)^{-n}]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2!}[(x-ia)^{-n} - (x+ia)^{-n}].$$

Putting $x = r\cos\theta$ and $a = r\sin\theta$, we have

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2!}[r^{-n}(\cos\theta - i\sin\theta)^{-n} - r^{-n}(\cos\theta + i\sin\theta)^{-n}]$$

$$= \frac{(-1)^{n-1}.(n-1)!}{2!} r^{-n}[(\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)]$$

$$= \frac{(-1)^{n-1}.(n-1)!}{2!} r^{-n}.2i\sin n\theta \qquad [\because \sin(-n\theta) = -\sin n\theta]$$

$$=(-1)^{n-1}.(n-1)!.r^{-n}\sin n\theta$$

$$=(-1)^{n-1}(n-1)!\left(\frac{a}{\sin\theta}\right)^{-n}\sin n\theta \qquad \left[\text{since } r=\frac{a}{\sin\theta}\right]$$

$$=(-1)^{n-1}(n-1)!a^{-n}\sin^n\theta.\sin n\theta.$$

Q.5. State and prove Leibnitz's theorem.

Ans. Leibnitz's Theorem : This theorem helps us to find the *n*th differential coefficient of the product of two functions. The statement of the theorem is as follows :

If u and v are any two functions of x such that all their desired differential coefficients exist, then the *n*th differential coefficient of their product is given by

$$D^n(uv)=(D^nu)\cdot v+{}^nC_1\,D^{n-1}u\cdot Dv+{}^nC_2\,D^{n-2}u\cdot D^2v+\ldots$$
$$\ldots+{}^nC_r\,D^{n-r}u\cdot D^rv+\ldots+u\,D^nv.$$

Proof : We shall prove the theorem by mathematical induction. By actual differentiation, we have

$$D(uv)=(Du)\cdot v+u\cdot Dv. \qquad \ldots(1)$$

From (1) we see that the theorem is true for $n=1$.

Now assume that the theorem is true for a particular value of n. Then we have

$$D^n(uv)=(D^nu)\cdot v+{}^nC_1\,D^{n-1}u\cdot Dv+{}^nC_2\,D^{n-2}u\cdot D^2v+\ldots$$
$$+{}^nC_r\,D^{n-r}u\cdot D^rv+{}^nC_{r+q}\,D^{n-r-1}u\cdot D^{r+1}v+\ldots+u\cdot D^nv. \quad \ldots(2)$$

Differentiating both sides of (2) with respect to x, we get

$$D^{n+1}(uv)=\{(D^{n+1}u)\cdot v\,D^nu\cdot Dv\}+\{{}^nC_1D^nu\cdot Dv+{}^nC_1\,D^{n-1}u\cdot D^2v\}$$
$$+\{{}^nC_2\,D^{n-1}u\cdot D^2u+{}^nC_2\,D^{n-2}u\cdot D^3v\}+\ldots$$
$$+\{{}^nC_r\,D^{n-r+1}u\cdot D^rv+{}^nC_r\,D^{n-r}u\cdot D^{r+1}v\}$$
$$+\{{}^nC_{r+1}\,D^{n-r}u\cdot D^{r+1}v+{}^nC_{r+1}\,D^{n-r-1}u\cdot D^{r+2}v\}+\ldots$$
$$+\{Du\,D^nv+u\,D^{n+1}v\}.$$

Rearranging the terms, we get

$$D^{n+1}(uv)=(D^{n+1}u)\cdot v+(1+{}^nC_1)(D^nu\,Dv)+({}^nC_1+{}^nC_2)D^{n-1}u\cdot D^2v$$
$$+\ldots+({}^nC_r+{}^nC_{r+1})(D^{n-r}u\cdot D^{r+1}v)+\ldots+u\,D^{n+1}v. \quad \ldots(3)$$

But we know that, ${}^nC_r+{}^nC_{r+1}={}^{n+1}C_{r+1}$.

Therefore $1+{}^nC_1={}^nC_0+{}^nC_1={}^{n+1}C_1,\ {}^nC_1+{}^nC_2={}^{n+1}C_2$, and so on.

Hence (3) become

$$D^{n+1}(uv)=(D^{n+1}u)\cdot v+{}^{n+1}C_1\,(D^nu)\cdot Dv+{}^{n+1}C_2\,(D^{n-1}u)\cdot(D^2v)+\ldots$$
$$+{}^{n+1}C_{r+1}\,D^{n-r}u\cdot D^{r+1}v+\ldots+u\cdot D^{n+1}v. \quad \ldots(4)$$

From (4) we see that if the theorem is true for any value of n, it is also true for the next value of n. But we have already seen that the theorem is true for $n=1$. Hence it must be true for $n=2$ and so for $n=3$, and so on. Thus the theorem is true for all positive integral values of n.

Q.6. If $y=[x+\sqrt{(1+x^2)}]^m$, find the value of then differential coefficient of y for $x=0$.

Ans. Here $y=[x+\sqrt{(1+x^2)}]^m$. ...(1)

$$\therefore \quad y_1=m[x+\sqrt{(1+x^2)}]^{m-1}\cdot\left[1+\frac{1}{2}\cdot\frac{2x}{\sqrt{(1+x^2)}}\right]$$

$$=\frac{m}{\sqrt{(1+x^2)}}[x+\sqrt{(1+x^2)}]^m=\frac{my}{\sqrt{(1+x^2)}} \qquad ...(2)$$

or $\quad y_1{}^2(1+x^2)-m^2y^2=0.$

Differentiating again, we get

$$2y_1y_2(1+x^2)+2xy_1{}^2-2m^2yy_1=0$$

or
$$y_2(1+x^2)+xy_1-m^2y=0, \qquad ...(3)$$

cancelling $2y_1$, since $2y_1\neq 0$.

Again differentiating (3) n times, we get

$$y_{n+2}(1+x^2)+{}^nC_1.2xy_{n+1}+{}^nC_2.2y_n+xy_{n+1}+{}^nC_1.y_n-m^2y_n=0$$

or
$$y_{n+2}(1+x^2)+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0. \qquad ...(4)$$

Putting $x=0$ in (1), (2), (3) and (4), we have

$$(y)_0=1,(y_1)_0=m,(y_2)_0=m^2 \text{ and } (y_{n+2})_0+(n^2-m^2).(y_n)_0=0$$

i.e., $\quad (y_{n+2})_0=(m^2-n^2).(y_n)_0.$ ...(5)

Case I : When n is odd.

Putting $n=1,3,5,7,\dots$ in (5), we get $(y_3)_0=(m^2-1^2)(y_1)_0=(m^2-1^2).m$,

$(y_5)_0=(m^2-3^2)(y_3)_0=(m^2-3^2)(m^2-1^2).m$, and so on.

Hence when n is odd, we have

$$(y_n)_0=\{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\dots(m^2-3^2)(m^2-1^2)m.$$

Case II : When n is even.

Putting $n=2,4,6,\dots$ in (5), we get

$(y_4)_0=(m^2-2^2)(y_2)_0=(m^2-2^2).m^2$,

$(y_6)_0=(m^2-4^2)(y_4)_0=(m^2-4^2)(m^2-2^2).m^2$, and so on.

Hence when n is even, we have

$$(y_n)_0=\{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\dots(m^2-2^2).m^2.$$

Q.7. Obtain by Maclaurin's theorem the first five terms in the expansion of $\log(1+\sin x)$.

Ans. Let $f(x)=\log(1+\sin x) \Rightarrow f(0)=0$

Then $f'(x)=\dfrac{\cos x}{1+\sin x} \Rightarrow f'(0)=1$

$$f''(x)=-\frac{\sin x}{1+\sin x}-\frac{\cos^2 x}{(1+\sin x)^2}=-\frac{\sin x}{1+\sin x}-[f'(x)]^2$$

$\Rightarrow$ $f''(0)=-[f'(0)]^2=-(1)^2=-1$

$$f'''(x)=-\frac{\cos x}{1+\sin x}+\frac{\sin x\cos x}{(1+\sin x)^2}-2f'(x)f''(x)$$

or
$$f'''(x)=-f'(x)+\frac{\sin x}{1+\sin x}f'(x)f''(x)$$
$$=-f'(x)+f'(x)[-f''(x)-(f'(x))^2]-2f'(x)f''(x)$$
$$f'''(x)=-f'(x)-[f'(x)]^3-3f'(x)f''(x)$$

$\Rightarrow$ $f'''(0)=-f'(0)-[f'(0)]^3-3f'(0)f''(0)=-1-(1)^3-3(1)(-1)=1$

$$f^{iv}(x)=-f''(x)-3[f'(x)]^2f''(x)-3[f''(x)]^2-3f'(x)f'''(x)$$
$$f^{iv}(0)=-f''(0)-3[f'(0)]^2f''(0)-3[f''(0)]^2-3f'(0)f'''(0)$$
$$=-(-1)-3(1)^2(-1)-3(-1)^2-3(1)(1)=-2$$
$$f^{v}(x)=-f'''(x)-3[f'(x)]^2f'''(x)-6f'(x)[f'''(x)]^2$$
$$-9f''(x)f'''(x)-3f'(x)f^{iv}(x)$$

$\Rightarrow$
$$f^{v}(0)=-f'''(0)-3[f'(0)]^2f'''(x)-6f'(x)[f''(0)]^2$$
$$-9f''(0)f'''(0)-3f'(0)f^{iv}(0)$$
$$=5$$

Now by Maclaurin's theorem (first five terms)

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\frac{x^5}{5!}f^{v}(0)$$
$$=0+x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{2x^4}{4!}+\frac{5x^5}{5!}=x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\frac{x^5}{24}$$

Q.8. Use Taylor's theorem to prove that

$$\tan^{-1}(x+h)=\tan^{-1}x+h\sin\theta\frac{\sin\theta}{1}-(h\sin\theta)^2\frac{\sin 2\theta}{2}$$
$$+(h\sin\theta)^3\frac{\sin 3\theta}{3}-\ldots+(-1)^{n-1}(h\sin\theta)^n\frac{\sin n\theta}{n}+\ldots,$$

where $\theta=\cot^{-1}x$.

Ans. Let $y = f(x) = \tan^{-1} x.$

Then $y_1 = \dfrac{1}{1+x^2} = \dfrac{1}{(x+i)(x-i)} = \dfrac{1}{2i}\left[\dfrac{1}{x-i} - \dfrac{1}{x+i}\right]$

or $y_1 = \dfrac{1}{2i}[(x-i)^{-1} - (x+i)^{-1}].$ ...(1)

Differentiating (1), $(n-1)$ times, we get

$$y_n = \frac{1}{2i}[(-1)^{n-1}(n-1)!(x-i)^{-n} - (-1)^{n-1}(n-1)!(x+i)^{-n}]$$

or $y_n = \dfrac{(-1)^{n-1}(n-1)!}{2i}[(x-i)^{-n} - (x+i)^{-n}].$ ...(2)

Now put $x = r\cos\theta, 1 = r\sin\theta$ in (2). Then

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}.r^{-n}[(\cos\theta - i\sin\theta)^{-n} - (\cos\theta + i\sin\theta)^{-n}]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}r^{-n}[(\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)],$$

by De Moivre's theorem

$$= \frac{(-1)^{n-1}(n-1)!}{2i}r^{-n}.2i\sin n\theta$$

$$= (-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta. \qquad [\because r^{-1} = 1/r = \sin\theta]$$

Hence $f^{(n)}(x) = (-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta,$

where $\cot\theta = x,$ *i.e.,* $\theta = \cot^{-1} x.$

Putting $n = 1, 2, 3, \ldots,$ we get

$$f'(x) = \sin\theta\cdot\sin\theta, f''(x) = -\sin^2\theta\sin 2\theta,$$

$$f'''(x) = 2!\sin^3\theta\sin 3\theta, \text{ and so on.}$$

Substituting these values in Taylor's series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x) + \ldots,$$

we get $\tan^{-1}(x+h) = \tan^{-1}x + h\sin\theta\cdot\sin\theta - \dfrac{h^2}{2!}\sin^2\theta\sin 2\theta$

$$+ \frac{h^3 2!}{3!}\sin^3\theta\sin 3\theta - \ldots + \frac{h^n}{n!}(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta + \ldots$$

or $\tan^{-1}(x+h) = \tan^{-1}x + h\sin\theta\cdot\dfrac{\sin\theta}{1} - (h\sin\theta)^2\dfrac{\sin 2\theta}{2}$

$$+ (h\sin\theta)^3.\frac{\sin 3\theta}{3} + \ldots + (-1)^{n-1}(h\sin\theta)^n.\frac{\sin n\theta}{n} + \ldots$$

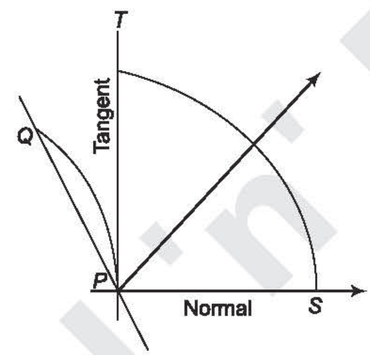
❑

UNIT III

SECTION-A (VERY SHORT ANSWER TYPE) QUESTIONS

Q.1. Define tangent and normal.

Ans. Tangent and Normal : Let P be a given point and Q be any other point on it. Let Q travel towards P along the curve. Then, the limiting position PT of the secant PQ is known as the **tangent** to the curve. The line PS through P which is perpendicular to the tangnent PT is called the **normal** of the curve.



Q.2. Define equation of the normal.

Ans. Equation of the Normal : The normal to a curve at a given point is a line perpendicular to the tangent at that point and passes through the point. The slope of the normal at point $P(x_1, y_1)$ will be negative reciprocal of the slope of the tangent.

Hence, the slope of the normal at $(x_1, y_1) = -\frac{1}{dy_1/dx_1}$.

$\therefore$ the equation of the normal at $P(x_1, y_1)$ is $y - y_1 = -\frac{1}{dy_1/dx_1}(x - x_1)$.

Q.3. Find the pedal equation of $r^n = a^n \sin n\theta$.

Ans. Hence, the given curve is $r^n = a^n \sin n\theta$. ...(1)

Taking logarithm of both the sides of (1), we get

$$n \log r = n \log a + \log \sin n\theta. \quad ...(2)$$

Differentiating w.r.t. θ, we get

$$\frac{n}{r} \cdot \frac{dr}{d\theta} = n \frac{\cos n\theta}{\sin n\theta} = n \cot n\theta$$

$$\Rightarrow \quad \cot\phi = \frac{1}{r} \cdot \frac{dr}{d\theta} = \cot n\theta. \quad \therefore \quad \phi = n\theta.$$

Also, $\quad p = r \sin\theta \quad \Rightarrow \quad p = r \sin n\theta,$

Now from (1) and (3), we have $\sin n\theta = \frac{p}{r}$.

Putting the value in (1), we get $pa^n = r^{n+1}$.

Q.4. Find the angle at which the radius vector cuts the curves $\frac{1}{r} = 1 + e\cos\theta$.

Ans. Here, the given equation of the curve is $\frac{1}{r} = 1 + e\cos\theta$.

$$\Rightarrow \qquad \log l - \log r = \log(1 + e\cos\theta)$$

Diff. w.r.t. θ, we get $-\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{1}{(1+e\cos\theta)}(-e\sin\theta)$.

$$\therefore \qquad \cot\phi = \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{e\sin\theta}{1+e\cos\theta}$$

$$\Rightarrow \qquad \tan\phi = \frac{1+e\cos\theta}{e\sin\theta} \quad \Rightarrow \quad \phi = \tan^{-1}\left[\frac{1+e\cos\theta}{e\sin\theta}\right]$$

Q.5. If the tangent to the curve $x^{1/2} + y^{1/2} = a^{1/2}$ at any point on it cuts the aces *OX, OY* at *P, Q* respectively, prove that $OP + OQ = a$.

Ans. The curve is $\left(\frac{x}{a}\right)^{1/2} + \left(\frac{y}{a}\right)^{1/2} = 1.$ ...(1)

The co-ordinates of any point (x, y) on (1) may be taken as

$$x = a\cos^4 t,\; y = a\sin^4 t.$$

$$\therefore \quad \frac{dx}{dt} = -4a\cos^3 t\sin t \quad \text{and} \quad \frac{dy}{dt} = 4a\sin^3 t\cos t. \quad \therefore \frac{dy}{dx} = -\frac{\sin^2 t}{\cos^2 t}.$$

The equation of the tangent at the point 't' to (1) is

$$(Y - a\sin^4 t) = -\frac{\sin^2 t}{\cos^2 t}(X - a\cos^4 t)$$

or $\qquad X\sin^2 t + Y\cos^2 t = a\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t)$

or $\qquad X\sin^2 t + Y\cos^2 t = a\sin^2 t\cos^2 t.$ ...(2)

Now (2) meets OX where $Y = 0$ *i.e.*, where

$$X\sin^2 t = a\sin^2 t\cos^2 t \text{ or } X = a\cos^2 t.$$

Therefore $\qquad OP = a\cos^2 t.$

Similarly, $\qquad OQ = a\sin^2 t.$

Hence $\qquad OP + OQ = a\cos^2 t + a\sin^2 t = a.$

Q.6. Find the asymptotes of the curve $y^2 = 4x$.

Ans. The equation of the curve is $y^2 - 4x = 0$.

Putting $y = m$ and $x = 1$ in the highest *i.e.*, 2nd degree terms, we get $\phi_2(m) = m^2$.

Solving the equation $\phi_2(m)=0$ *i.e.,* $m^2=0$, we get $m=0,0$.

Also putting $y=m$ and $x=1$ in the first degree terms, we get $\phi_1(m)=-4$.
Now c is given by the equation $c\phi'_2(m)+\phi_1(m)=0$ *i.e.,* $2mc-4=0$.
If we put $m=0$ in this equation, we get $c=\infty$. Hence no asymptote exists.

Q.7. Define non-algebraic curve.

Ans. A curve in which there are some terms involving cosine, sine etc., is called non-algebraic curve.

Q.8. Find the radius of curvature for the curve whose intrinsic equation is $s=a\log\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)$.

Ans. We have $\rho=\frac{ds}{d\psi}=a\frac{1}{\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}\sec^2\left(\frac{\pi}{4}+\frac{\psi}{2}\right).\frac{1}{2}$

$$=\frac{a}{2\sin\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\cos\left(\frac{\pi}{4}+\frac{\psi}{2}\right)}=\frac{a}{\sin\left(\frac{\pi}{2}+\psi\right)}=\frac{a}{\cos\psi}=a\sec\psi.$$

Q.9. Find the radius of curvature at the point (p,r) on the ellipse

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}.$$

Ans. Differentiating the given equation with respect to r, we get

$$-\frac{2}{p^3}\frac{dp}{dr}=-\frac{2r}{a^2b^2}\quad\text{or}\quad\frac{dp}{dr}=\frac{rp^3}{a^2b^2}$$

Hence $\rho=r\frac{dr}{dp}=r.\frac{a^2b^2}{r\,p^3}=\frac{a^2b^2}{p^3}.$

Q.10. Find the nature of the origin on the curve $a^4y^2=x^4(x^2-a^2)$.

Ans. The given curve is $a^4y^2=x^4(x^2-a^2)$. ...(1)

Equating to zero the lowest degree terms in the equation of the curve, we get the tangents at the origin as $a^4y^2=0$ *i.e.,* $y=0$, $y=0$ are two real and coincident tangents at the origin.

Thus the origin may be a cusp or a conjugate point.

From (1), $y=\pm(x^2/a^2)\sqrt{(x^2-a^2)}$.

For small values of $x\neq0$, +ive or –ive, (x^2-a^2) is –ive *i.e.,* y is imaginary. Hence no portion of the curve lies in the neighbourhood of the origin. Hence origin is a conjugate point and not a cusp.

Q.11. Write the definition of Evolute.

Ans. The evolute of a curve is the envelope of the normals to that curve. In other words, the locus of the centre of curvature of a curve is called evolute for that curve.

SECTION-B (SHORT ANSWER TYPE) QUESTIONS

Q.1. Find the equation of the tangent and the normal at the point 't' to the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$.

Ans. Here, we have $x = a(t + \sin t)$, $y = a(t - \cos t)$

then, $\dfrac{dy}{dx} = a(1 + \cos t)$, $\dfrac{dy}{dt} = a \sin t$

Now, $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1 + \cos t)} = \frac{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}{2\cos^2\left(\frac{t}{2}\right)} = \tan\left(\frac{t}{2}\right)$$

The equation of tangent at 't' point is

$$y - a(1 - \cot t) = \tan\left(\frac{t}{2}\right)[x - a(t + \sin t)]$$

$$\Rightarrow \quad y - 2a\sin^2\left(\frac{t}{2}\right) = (x - at)\tan\left(\frac{t}{2}\right) - a \sin t \,.\, \tan\left(\frac{t}{2}\right)$$

$$\Rightarrow \quad y - 2a\sin^2\left(\frac{t}{2}\right) = (x - at)\tan\left(\frac{t}{2}\right) - 2a\sin^2\left(\frac{t}{2}\right) \quad \Rightarrow \quad y = (x - at)\tan\left(\frac{t}{2}\right).$$

Again the equation of the normal at the point 't' is

$$y - a(1 - \cos t) = -\frac{1}{\tan\frac{1}{2}t}\{x - a(t + \sin t)\}$$

or $$\left(y - 2a\sin^2\frac{1}{2}t\right)\tan\frac{1}{2}t = -x + a(t + \sin t)$$

or $$x + y\tan\frac{1}{2}t = a\left(t + \sin t + 2\sin^2\frac{1}{2}t\tan\frac{1}{2}t\right).$$

Q.2. Find the angles of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

Ans. The given curves are $y^2 = 4ax$, ...(1)

and $x^2 = 4by$. ...(2)

Solving (1) and (2), we get on eliminating y

$$x^4 = 64ab^2x \quad \text{or} \quad x(x^3 - 64ab^2) = 0.$$

$$\therefore \quad x = 0 \quad \text{and} \quad 4a^{1/3}b^{2/3}.$$

Substituting these values of x in (2), we get

$$y = 0 \text{ for } x = 0 \text{ and } y = 4a^{2/3}b^{1/3} \text{ for } x = 4a^{1/3}b^{2/3}.$$

Therefore $(0, 0)$ and $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$ are the two points of intersection of (1) and (2).

Differentiating (1), we get $2y\frac{dy}{dx}=4a$ i.e., $\frac{dy}{dx}=\frac{2a}{y}$.

Differentiating (2), we get $2x=4b\frac{dy}{dx}$ i.e., $\frac{dy}{dx}=\frac{x}{2b}$.

Angle of intersection at (0, 0).

$$\frac{dy}{dx} \text{ of (1) at } (0,0)=\infty \quad \text{and} \quad \left(\frac{dy}{dx}\right) \text{ of (2) at } (0,0)=0.$$

∴ The angle of intersection at (0, 0) is 90°.

Angle of intersection at $(4a^{1/3}\,b^{2/3}, 4a^{2/3}\,b^{1/3})$.

$$\frac{dy}{dx} \text{ of (1) at } (4a^{1/3}\,b^{2/3}, 4a^{2/3}\,b^{1/3})=\frac{a^{1/3}}{2b^{1/3}},$$

and
$$\frac{dy}{dx} \text{ of (2) at } (4a^{1/3}\,b^{2/3}, 4a^{2/3}\,b^{1/3})=\frac{2a^{1/3}}{b^{1/3}}.$$

Therefore if θ is the acute angle between the tangents to the two curves at the point $(4a^{1/3}\,b^{2/3}, 4a^{2/3}\,b^{1/3})$, then

$$\theta=\tan^{-1}\left|\frac{\frac{2a^{1/3}}{b^{1/3}}-\frac{a^{1/3}}{2b^{1/3}}}{1+\frac{2a^{1/3}}{b^{1/3}}\frac{a^{1/3}}{2b^{1/3}}}\right|=\tan^{-1}\frac{3a^{1/3}\,b^{1/3}}{2\,(a^{2/3}+b^{2/3})}.$$

Q.3. Show that the pedal equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}.$$

Ans. Here, the equation of the curve is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let $x=a\cos t,\ y=b\sin t$.

$$\therefore \quad \frac{dx}{dt}=-a\sin t,\ \frac{dy}{dt}=b\cos t \quad \Rightarrow \quad \frac{dy}{dx}=-\frac{b\cos t}{a\sin t}.$$

Therefore, the equation of the tangent at '*t*' is

$$Y-b\sin t=-\frac{b\cos t}{a\sin t}(X-a\cos t)$$

$$\Rightarrow \quad ab-b\cos t\,.\,X-a\sin t\,.\,Y=0. \qquad \text{...(1)}$$

Since p denote the length perpendicular from (0, 0) to (1)

$$\therefore \quad p=\frac{ab}{\sqrt{a^2\sin^2 t+b^2\cos^2 t}}$$

$$\Rightarrow \qquad \frac{1}{p^2}=\frac{a^2\sin^2 t+b^2\cos^2 t}{a^2b^2} \qquad ...(2)$$

Now, $r^2=x^2+y^2=a^2\cos^2 t+b^2\sin^2 t=a^2+b^2-a^2\sin^2 t-b^2\cos^2 t.$ ...(3)

From (3) $\quad a^2\sin^2 t+b^2\cos^2 t=(a^2+b^2)-r^2$

Therefore, from (3), we get

$$\frac{1}{p^2}=\frac{(a^2+b^2)-r^2}{a^2b^2}=\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}.$$

Q.4. Show that the parabolas $r=a/(1+\cos\theta)$ and $r=b/(1-\cos\theta)$ intersect orthogonally.

Ans. The given curves are

$$r=a/(1+\cos\theta) \qquad ...(1)$$

and

$$r=b/(1-\cos\theta). \qquad ...(2)$$

Taking logarithm of both sides of (1), we get

$$\log r=\log a-\log(1+\cos\theta).$$

Differentiating both sides w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta}=-\frac{(-\sin\theta)}{1+\cos\theta}=\frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}=\tan\frac{1}{2}\theta.$$

$$\therefore \qquad \cot\phi_1=\tan\frac{1}{2}\theta=\cot\left(\frac{1}{2}\pi-\frac{1}{2}\theta\right) \text{ or } \phi_1=\pi/2/2-\theta/2.$$

Hence $\quad \phi_1=\frac{1}{2}\pi-\frac{1}{2}\theta.$

Again taking logarithm of both sides of (2), we get

$$\log r=\log b-\log(1-\cos\theta).$$

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{-\sin\theta}{1-\cos\theta}=\frac{-2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta}=-\cot\frac{1}{2}\theta.$$

$$\therefore \qquad \cot\phi_2=-\cot\frac{1}{2}\theta=\cot\left(\pi-\frac{1}{2}\theta\right) \text{ or } \phi_2=\pi-\frac{1}{2}\theta.$$

Hence $\quad \phi_2=\pi-\frac{1}{2}\theta.$

Therefore angle of intersection $=\phi_1\sim\phi_2=\left(\pi-\frac{1}{2}\theta\right)-\left(\frac{1}{2}\pi-\frac{1}{2}\theta\right)=\frac{1}{2}\pi.$

Thus the two curves intersect orthogonally.

Q.5. In the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$, prove that $\rho = 4a \cos \frac{t}{2}$.

Ans. We have $x = a(t + \sin t) \Rightarrow \frac{dx}{dt} = a(1 + \cos t)$

$$y = a(1 - \cos t) \Rightarrow \frac{dy}{dt} = a \sin t$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1+\cos t)} = \frac{2 \sin t/2 \cos t/2}{2\cos^2 t/2} = \tan t/2.$$

Also $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\tan\frac{t}{2}\right) = \frac{1}{2}\sec^2\frac{t}{2}\cdot\frac{dt}{dx}.$$

$$\frac{1}{2}\sec^2\frac{t}{2}\cdot\frac{1}{a(1+\cos t)} = \frac{1}{4a}\sec^4\frac{t}{2}.$$

Now, putting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in $\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$.

We get $$\rho = \frac{[1 + \tan^2 t/2]^{3/2}}{\frac{1}{4a}\sec^4 t/2} = \frac{4a \sec^3 t/2}{\sec^4 t/2} = 4a \cos t/2.$$

Q.6. Prove that for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, radius of curvature $\rho = \frac{a^2b^2}{p^3}$ ρ being the perpendicular from centre upon the tangent at (x, y).

Ans. We have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{b^2x}{a^2y} \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left[\frac{y - x\frac{dy}{dx}}{y^2}\right] = -\frac{b^4}{a^2y^3}.$$

Let $(a\cos\theta, b\sin\theta)$ be any point on the ellipse. The equation of the tangent at this point is

$$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta) \quad \text{or} \quad bx\cos\theta + ay\sin\theta - ab = 0. \qquad \text{...(2)}$$

We are given that

p = perpendicular from (0, 0) to the tangent (2)

or $$p = -\frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \qquad \text{...(3)}$$

Now the radius of curvature ρ is

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}=\frac{a^2y^2\left(1+\frac{b^4x^2}{a^4y^2}\right)}{-b^4}=\frac{(a^4y^2+b^4x^2)^{3/2}}{-a^4b^4}$$

Then ρ at $(a\cos\theta, b\sin\theta)$ is

$$\rho=-\frac{(a^4b^2\sin^2\theta+b^4a^2\cos^2\theta)^{3/2}}{a^4b^4}=-\frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}{ab}$$

$$=-\frac{(-ab/p)^3}{ab}$$ [Using (3)]

$$\rho=\frac{a^2b^2}{p^3}$$

Q.7. Show that for the cardioid $r=a(1+\cos\theta)$, $\rho=\frac{2}{3}\sqrt{(2ar)}$.

Ans. The curve is $r=a(1+\cos\theta)$.

$$\therefore\quad \frac{dr}{d\theta}=-a\sin\theta \quad\text{and}\quad \frac{d^2r}{d\theta^2}=-a\cos\theta.$$

Now $$\rho=\frac{\{r^2+(dr/d\theta)^2\}^{3/2}}{r^2+2(dr/d\theta)^2-r(d^2r/d\theta^2)}$$

$$=\frac{\{a^2(1+\cos\theta)^2+a^2\sin^2\theta\}^{3/2}}{a^2(1+\cos\theta)^2+2(-a\sin\theta)^2-a(1+\cos\theta)(-a\cos\theta)}$$

$$=\frac{\left(4a^2\cos^4\frac{1}{2}\theta+4a^2\cos^2\frac{1}{2}\theta\sin^2\frac{1}{2}\theta\right)^{3/2}}{a^2+2a^2(\cos^2\theta+\sin^2\theta)+3a^2\cos\theta}$$

$$=\frac{(4a^2\cos^2\frac{1}{2}\theta)^{3/2}[\cos^2\frac{1}{2}\theta+\sin^2\frac{1}{2}\theta]^{3/2}}{3a^2(1+\cos\theta)}=\frac{8a^3\cos^3\frac{1}{2}\theta}{6a^2\cos^2\frac{1}{2}\theta}=\left[\frac{4a}{3}\right]\cos\frac{1}{2}\theta.$$

But $r=a(1+\cos\theta)=2a\cos^2\frac{1}{2}\theta$. $\quad\therefore\quad \cos\frac{1}{2}\theta=\sqrt{(r/2a)}$.

Hence $\rho=\frac{4a}{3}\sqrt{\left[\frac{r}{2a}\right]}=\frac{2}{3}\sqrt{(2ar)}$.

Q.8. Find the radius of curvature at the point (r, θ) of the curve

$$\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right)$$

Ans. The given equation is

$$\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right)$$

$$\Rightarrow \quad \frac{d\theta}{dr} = \frac{1}{a} \cdot \frac{1}{2} \cdot \frac{2r}{\sqrt{r^2 - a^2}} + \frac{1}{\sqrt{1 - (a/r)^2}}\left(-\frac{a}{r^2}\right)$$

$$= \frac{r}{a\sqrt{a^2 - r^2}} - \frac{a}{r\sqrt{r^2 - a^2}} = \frac{r^2 - a^2}{ar\sqrt{r^2 - a^2}} = \frac{\sqrt{r^2 - a^2}}{ar}$$

$$\Rightarrow \quad \frac{dr}{d\theta} = \frac{ar}{\sqrt{r^2 - a^2}} \qquad \text{...(1)}$$

Again, differentiating w.r.t. θ, we get

$$\frac{d^2r}{d\theta^2} = \frac{\left(\sqrt{r^2 - a^2}\right).a - ar.\dfrac{2r}{2\sqrt{r^2 - a^2}}}{r^2 - a^2} \cdot \frac{dr}{d\theta}$$

$$= \frac{a(r^2 - a^2) - ar^2}{(r^2 - a^2)^{3/2}} \times \frac{ar}{\sqrt{r^2 - a^2}} = -\frac{a^4 r}{(r^2 - a^2)^2} \qquad \text{...(2)}$$

Hence
$$\rho = \frac{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}} \qquad \text{...(3)}$$

Using (1) and (2) in (3) and then simplifying, we get

$$\rho = \frac{r^6 (r^2 - a^2)^{1/2}}{r^6} = (r^2 - a^2)^{1/2}$$

Q.9. Find the envelope of the family of circles $x^2 + y^2 - 2ax\cos\alpha - 2ay \sin\alpha = c^2$, where α being the parameter, and interpret the result.

Ans. The equation of the family of circles is

$$x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha = c^2. \qquad \text{...(1)}$$

Differentiating (1) partially w.r.t. α, we get

$$2ax\sin\alpha - 2ay\cos\alpha = 0 \quad \text{or} \quad x\sin\alpha - y\cos\alpha = 0 \qquad \text{...(2)}$$

Eliminating α between (1) and (2), we get

$$4a^2(x\sin\alpha - y\cos\alpha)^2 + 4a^2(x\cos\alpha + y\sin\alpha)^2 = 0 + (x^2 + y^2 - c^2)^2$$

or
$$4a^2(x^2 + y^2) = (x^2 + y^2 - c^2)^2.$$

This is the required envelope.

Interpretation : The equation of envelope can be written as

$$(x^2+y^2)^2-(4a^2+2c^2)(x^2+y^2)+c^4=0$$

It is quadratic in (x^2+y^2) so solving, we get

$$x^2+y^2=\frac{2(2a^2+c^2)\pm\sqrt{4(2a^2+c^2)^2-4c^4}}{2}$$

$$=2a^2+c^2\pm 2a\sqrt{c^2+a^2}=(\sqrt{a^2+c^2}\pm a)^2.$$

Thus, the equation of the envelope contains two circles with centred at (0, 0) and radius

$$\sqrt{a^2+c^2}\pm a.$$

Q.10. Show that the evolute of an equiangular spiral is an equiangular spiral.

Ans. The pedal equation of an equiangular spiral is

$$p=r\sin\alpha \quad ...(1)$$

so that $\frac{dp}{dr}=\sin\alpha.$

$\therefore$ $\rho=r\frac{dr}{dp}=r.\frac{1}{\sin\alpha}=r\operatorname{cosec}\alpha$ or $\rho=r\operatorname{cosec}\alpha.$ ...(2)

Let (p', r') be any point on the evolute corresponding to the point (p, r) on the curve (1). Then we have,

$$r'^2=r^2+\rho^2-2\rho p$$

$$=r^2+r^2\operatorname{cosec}^2\alpha-2r\operatorname{cosec}\alpha.r\sin\alpha \quad \text{[Using (1) and (2)]}$$

$$=r^2\operatorname{cosec}^2\alpha-r^2$$

$$r'^2=r^2+\cot^2\alpha. \quad ...(3)$$

Also, we have $p'^2=r^2-p^2=r^2-r^2\sin^2\alpha=r^2(1-\sin^2\alpha)$

$$p'^2=r^2\cos^2\alpha. \quad ...(4)$$

Dividing (4) by (3), we get

$\therefore$ $\frac{p'^2}{r'^2}=\frac{r^2\cos^2\alpha}{r^2\cot^2\alpha}=\sin^2\alpha.$

$p'^2=r'^2\sin^2\alpha$ or $p'=r'\sin\alpha.$

Thus the locus of the point (p', r') is $p=r\sin\alpha$, which is an equiangular spiral.

Q.11. Find the points of inflexion of the curve $y^2=x(x+1)^2$.

Ans. The equation of the curve can be written as $y=(x+1)\sqrt{x}$. ...(1)

Differentiating (1) w.r.t. 'x', we get $\frac{dy}{dx}=\frac{3}{2}.x^{1/2}+\frac{1}{2\sqrt{x}}.$

Again differentiating w.r.t. 'x' $\frac{d^2y}{dx^2}=\frac{3}{4\sqrt{x}}-\frac{1}{4x^{3/2}}$ ...(2)

and again differentiating w.r.t. 'x', we get $\frac{d^3y}{dx^2}=-\frac{3}{8x^{3/2}}+\frac{3}{8x^{5/2}}.$ ...(3)

for the point of inflexion, we have $\dfrac{d^2y}{dx^2}=0.$

$$\therefore \qquad \frac{3}{4\sqrt{x}}-\frac{1}{4x\sqrt{x}}=0 \quad \text{or} \quad \left(3-\frac{1}{x}\right)=0 \quad \text{or} \quad x=\frac{1}{3}.$$

From (3) it is obvious that at $x=\dfrac{1}{3},\dfrac{d^3y}{dx^3}\neq 0.$ Thus, the point of inflexion are given by $\left(\dfrac{1}{3},\pm 4/3\sqrt{3}\right).$

Q.12. Show that the sine curve $y=\sin x$ is everywhere concave with respect to the axis of x excluding the points where it meets the axis of x.

Ans. The given curve is $y=\sin x$.

We have $\dfrac{dy}{dx}=\cos x$ and $d^2\dfrac{y}{dx^2}=-\sin x.$

The function $\sin x$ is a periodic function with period 2π. Hence, it is suifficient to consider the given curve in the interval $[0,2\pi]$.

In the interval $[0,2\pi]$, we have $y=0$ when $x=0$ or $x=\pi$ or $x=2\pi$.

When $x\in\,]0,\pi[$, we have $y>0$ and $\dfrac{d^2y}{dx^2}>0.$

So $y\dfrac{d^2y}{dx^2}$ when $x\in\,]\pi,2\pi[$.

Hence, the curve $y=\sin x$ is concave to the axis of x in the interval $]\pi,2\pi[$.

Thus the curve $y=\sin x$ is everywhere concave with respect to the axis of x excluding the points where it meets the axis of x.

Q.13. Find the points of inflexion on the curve $r(\theta^2-1)=a\theta^2$.

Ans. We have $r=a\theta^2/(\theta^2-1)$.

$$\therefore \qquad \frac{dr}{d\theta}=a\,[(\theta^2-1).2\theta-\theta^2.2\theta]/(\theta^2-1)^2=-2a\theta/(\theta^2-1)^2,$$

and

$$\frac{d^2r}{d\theta^2}=-2a\,[(\theta^2-1)^2.1-\theta.2(\theta^2-1).2\theta]/(\theta^2-1)^4=2a\,(3\theta^2+1)/(\theta^2-1)^3.$$

We know that at the point of inflexion, the radius of curvature is infinite. Hence at the point of inflexion, we have

$$r^2+2(dr/d\theta)^2-r\,(d^2r/d\theta^2)=0$$

or

$$\frac{a^2\theta^4}{(\theta^2-1)^2}+\frac{8a^2\theta^2}{(\theta^2-1)^4}-\frac{2a^2\theta^2(3\theta^2+1)}{(\theta^2-1)^4}=0$$

or $$\frac{a^2\theta^2(\theta^2-3)(\theta^2+2)}{(\theta^2-1)^3}=0$$

or $$\theta^2(\theta^2-3)(\theta^2+2)=0$$

$\therefore$ $\theta^2=0,3,-2.$

Rejecting the values $\theta^2=-2$ and 0 we see that the points of inflexion are given by

$$\theta^2=3 \text{ i.e., } \theta=\pm\sqrt{3}.$$

Q.14. Examine the curve $y=\sin x$ for concavity upwards, concavity downwards and for points of inflexion in the interval $[-2\pi, 2\pi]$.

Ans. The given curve is $y=\sin x$.

We have $\frac{dy}{dx}=\cos x$, $\frac{d^2y}{dx^2}=-\sin x$ and $\frac{d^3y}{dx^3}=-\cos x$.

Also, we have $\frac{d^2y}{dx^2}<0,\ \forall\ x\in\,]-2\pi,-\pi[$,

$$\frac{d^2y}{dx^2}>0,\ \forall\ x\in\,]-\pi,0[,\ \frac{d^2y}{dx^2}<0,\ \forall\ x\in\,]0,\pi[$$

and $$\frac{d^2y}{dx^2}>0,\ \forall\ x\in\,]\pi,2\pi[.$$

Hence, the curve is concave downwards in the intervals $]-2\pi,-\pi[$ and $0,\pi$ [and concave upwards in the intervals] $]-\pi,0[$ and $]\pi,2\pi[$.

Now $\frac{d^2y}{dx^2}=0 \Rightarrow \sin x=0 \Rightarrow x=-2\pi$ or $x=-\pi$ or $x=0$ or $x=\pi$ or $x=2\pi$

At each of the points $x=-2\pi, x=-\pi, x=0, x=\pi$ and $x=2\pi$, we have $\frac{d^3y}{dx^3}\neq 0$.

Thus there are points of inflexion at each of these points.

Also, $y=0$ at each of these points.

Hence, the curve has points of inflexion at $(-2\pi,0)$, $(-\pi,0)$, $(0,0)$ $(\pi,0)$ and $(2\pi,0)$.

Q.15. Trace the curve $x=a(t-\sin t)$, $y=a(1-\cos t)$.

Ans. We have $x=a(t-\sin t)$, $\Rightarrow \frac{dx}{dt}=a(1-\cos t)$.

$$y=a(1-\cos t),\ \Rightarrow \frac{dy}{dt}=a\sin t.$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a\sin t}{a(1-\cos t)}=\frac{2\sin t/2\cos t/2}{2\sin^2 t/2}=\cot\frac{t}{2}$$

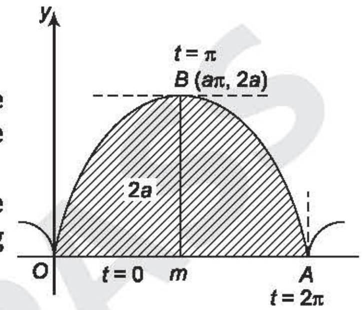
Here, $y=0$ when $\cos t=1$, *i.e.*, $t=0, 2\pi$. When $t=0, x=0$, and $\frac{dy}{dx}=\cot 0=\infty$. Therefore the curve passes through the origin and axis of y is tangent to the curve at this point.

Also y is maximum when $\cos t=-1$, *i.e.*, $t=\pi$.

When $t=\pi, x=a(\pi-\sin\pi)=a\pi, y=2a, \frac{dy}{dx}=\cot\frac{\pi}{2}=0.$

Therefore at $t=\pi$, whose Cartesian coordinates are $(a\pi, 2a)$ the tangent to the curve is parallel to x-axis and curve does not lie in the region $y>2a$.

In this curve y cannot be negative because $\cos t$ cannot be greater than 1. Hence one complete arc of the region cycloid lying between $0\le t\le 2\pi$.



Q.16. Trace the curve $r=a+b\cos\theta, a<b$.

Ans. (i) The curve is symmetrical about the initial line.

(ii) $r=0$, when $a+b\cos\theta=0$ *i.e.*, $\cos\theta=\left(-\frac{a}{b}\right)$ or $\theta=\cos^{-1}\left(-\frac{a}{b}\right)$ but $a<b$, *i.e.*, $\frac{a}{b}<1$, therefore $\cos^{-1}\left(-\frac{a}{b}\right)$ comes out of be real so that $\theta=\cos^{-1}\left(-\frac{a}{b}\right)$ is the tangent at the pole.

(iii) r is maximum when $\cos\theta=1$, *i.e.*, $\theta=0$.

Then the maximum value of $r=a+b$ and the minimum value of $r=a-b$ when $\cos\theta=-1$ *i.e.*, $\theta=\pi$.

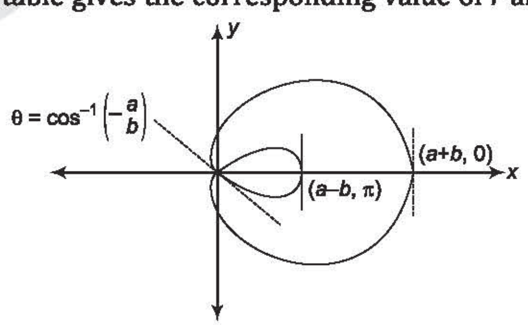
(iv) Since we have $\quad r=a+b\cos\theta.$

$$\therefore \qquad \frac{dr}{d\theta}=-b\sin\theta$$

then
$$\tan\phi=r\frac{d\theta}{dr}=-\frac{(a+b)\cos\theta}{b\sin\theta}.$$

Now if $\theta=0$ and $\pi, \phi=90°$, thus at the points $(a+b, 0), (a-b, \pi)$ the tangents are perpendicular to the initial line.

(v) The following table gives the corresponding value of r and θ



θ	0	$\pi/2$	$\cos^{-1}\left(-\frac{a}{b}\right)$	$\cos\left(\frac{a}{b}\right)<\theta<\pi$	π
r	$a+b$	a	0	r is negative	$a-b$

SECTION-C LONG ANSWER TYPE QUESTIONS

Q.1. Find all the asymptotes of the curve $x^3 + x^2 y - xy^2 - y^3 - 3x, y - 1 = 0$.

Ans. The degree of the curve is 3 so it has 3 asymptotes which are real as well as imaginary. Since the coefficients of highest degree, *i.e.,* 3rd degree of x and y are constant so there are no asymptotes parallel to co-ordinate axes. Thus there are **oblique asymptotes** of the form $y = mx + c$.

Now putting $y = m$ and $x = 1$ in the third degree terms of the curve, we get

$$\phi_3(m) = 1 + m - m^2 - m^3.$$

Solving the equation $\phi_3(m) = 0$ *i.e.,* $1 + m - m^2 - m^3 = 0$, we get

$$(1+m)(1-m^2) = 0 \quad \text{or} \quad m = -1, -1, 1.$$

Determination of *c*. For $m = 1$, we use the following equation

$$c\,\phi_n'(m) + \phi_{n-1}(m) = 0 \text{ or } c\,\phi_3'(m) + \phi_2(m) = 0. \quad ...(1)$$

Putting $y = m$ and $x = 1$ in the second degree terms of the equation we get

$$\phi_2(m) = 0.$$

From (1), we get $\quad c[1 - 2m - 3m^2] + 0 = 0$

at $m = 1$ $\quad c(1 - 2 - 3) + 0 = 0 \quad$ or $\quad -4c = 0 \quad$ or $\quad c = 0.$

Thus one of the asymptote is $y = x$.

Determination of *c* for $m = -1, -1$. Since two out of three roots of the equation $\phi_3(m) = 0$ are same, then we use the following formula to determine c

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0. \quad ...(2)$$

Putting $y = m$ and $x = 1$ in the first degree terms of the equation we obtain $\phi_1(m) = -3 - m$.

From (2), we have

$$\frac{c^2}{2!}(-2 - 6m) + \frac{c}{1!}.0 + (-3 - m) = 0$$

at $m = -1$ $\quad \dfrac{c^2}{2}(-2 + 6) - 3 + 1 = 0 \quad$ or $\quad 2c^2 - 2 = 0 \quad$ or $\quad c = \pm 1.$

Thus other two asymptotes are

$$y = -x + 1, y = -x - 1.$$

Hence, all the asymptotes of the given curve are

$$y = x, x + y - 1 = 0, x + y + 1 = 0.$$

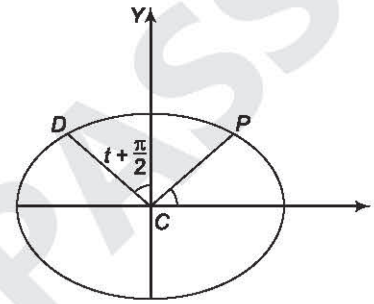
Q.2. If *CP, CD* be a pair of conjugate semi-diameters of an ellipse, show that the radius of curvature at *P* is $\dfrac{CD^3}{ab}$ where *a* and *b* are the lengths of the semi axis of the ellipse.

Ans. Let CP and CD be a conjugate of semi-diameters of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$

Let the coordinate of P are $x=a\cos t,\; y=b\sin t.$ ...(1)

Also, the coordinate of D are

$$\left[a\cos\left(\frac{\pi}{2}+t\right), b\sin\left(\frac{\pi}{2}+t\right)\right]=(-a\sin t, b\cos t).$$



From (1), we have

$$\frac{dx}{dt}=-a\sin t, \frac{dy}{dt}=b\cos t$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=-\frac{b}{a}\cot t$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\frac{b}{a}\cot t\right)$$

$$=\frac{d}{dt}\left(-\frac{b}{a}\cot t\right)\frac{dt}{dx}=\left(\frac{b}{a}\operatorname{cosec}^2 t\right)\left(-\frac{1}{a}\operatorname{cosec} t\right)=-\frac{b}{a^2}\operatorname{cosec}^3 t.$$

Putting the values of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{-\frac{b}{a^2}\operatorname{cosec}^3 t}=\frac{\left[1+\frac{b^2}{a^2}\cdot\frac{\cos^2 t}{\sin^2 t}\right]^{3/2}}{-\frac{b}{a^2}\operatorname{cosec}^3 t}=-\frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab}$$

$$\Rightarrow \quad \rho=\frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab}$$ (By neglecting the negative sign)

...(2)

From figure

$$CD=\sqrt{(-a\sin t-0)^2+(b\cos t-0)^2}=(a^2\sin^2 t+b^2\cos^2 t)^{1/2}.$$

$$\therefore \quad \frac{CD^3}{ab}=\frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab}.$$ ...(3)

Now from (2) and (3), we have $\rho=\dfrac{CD^3}{ab}.$

Q.3. Show that the circle of curvature at the origin of the parabola $y=mx+\dfrac{x^2}{a}$ is $(x^2+y^2)=a(1+m^2)(y-mx).$

Ans. We have $y=mx+\dfrac{x^2}{a}$

Differentiating eqn. (1) w.r. to *x*, we get

$$y' = m + \frac{2x}{a} \quad \text{and} \quad y'' = \frac{2}{a}$$

Now, $$\rho = \frac{(1+y'^2)^{3/2}}{y''} = \frac{a}{2}\left[1+\left(m+\frac{2x}{a}\right)^2\right]^{3/2}$$

At the points (0, 0), we get

$$\rho_0 = \frac{a}{2}[1+m^2]^{3/2}$$

Since, $$\Delta = \frac{1+y'^2}{y''} = \frac{1+\left(m+\frac{2x}{a}\right)^2}{2/a}$$

Assume that (α, β) is the centre of curvature at (x, y) then

$$\alpha = x - \Delta y' = x - \frac{a}{2}\, m + \frac{2x}{a}\left[1+\left(m+\frac{2x}{a}\right)^2\right]^2 \text{ and } \beta = y + \Delta = y + \frac{a}{2}\left[1+\left(m+\frac{2x}{a}\right)^2\right]$$

If (α_0, β_0) is the centre of curvature at (0, 0). Then we get

$$\alpha_0 = -\frac{ma}{2}(1+m^2) \text{ and } \beta_0 = -\frac{a}{2}(1+m^2)$$

Therefore, equation of circle of curvature is :

$$(x-\alpha_0)^2 + (y-\beta_0)^2 = \rho_0^2$$

$$\Rightarrow \quad \left[x+\frac{ma}{2}(1+m^2)\right]^2 + \left[y-\frac{a(1+m^2)}{2}\right]^2 = \frac{a^2}{4}(1+m^2)^3$$

$$\Rightarrow \quad x^2 + y^2 + ma(1+m^2)x - a(1+m^2)y + \frac{m^2a^2}{4}(1+m^2)^2 + \frac{a^2}{4}(1+m^2)^2 = \frac{a^2}{4}(1+m^2)^3$$

$$\Rightarrow \quad x^2 + y^2 + ma(1+m)^2 x - a(1+m^2)y = 0$$

$$\Rightarrow \quad x^2 + y^2 = a(1+m)^2[y - mx]$$

Q.4. Find the envelope of the family of curves $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$, where the parameters *a* and *b* are connected by the relation $a^p + b^p = c^p$.

Ans. The equation of the given family of curves is

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1, \qquad \text{...(1)}$$

where the parameters a and b are connected by the relation

$$a^p + b^p = c^p. \quad \text{...(2)}$$

Since there is a relation between a and b, therefore we shall regard b as a function of a. Now we shall differentiate (1) and (2) with respect to a regarding x and y as constants and b as a function of a.

From (1), we get

$$-\frac{mx^m}{a^{m+1}} - \frac{my^m}{b^{m+1}}\frac{db}{da} = 0 \quad \text{i.e.,} \quad \frac{db}{da} = -\frac{x^m/a^{m+1}}{y^m/b^{m+1}}. \quad \text{...(3)}$$

Again from (2), we get

$$pa^{p-1} + pb^{p-1}(db/da) = 0$$

i.e., $$db/da = -a^{p-1}/b^{p-1}. \quad \text{...(4)}$$

Equating the two values of (db/da), we get

$$\frac{x^m/a^{m+1}}{y^m/b^{m+1}} = \frac{a^{p-1}}{b^{p-1}} \quad \text{or} \quad \frac{x^m/a^m}{y^m/b^m} = \frac{a^p}{b^p}. \quad \text{...(5)}$$

Eliminating a and b between (1), (2) and (5), we get the required envelope. From (5), we have

$$\frac{x^m/a^m}{a^p} = \frac{y^m/b^m}{b^p} = \frac{x^m/a^m + y^m/b^m}{a^p + b^p} = \frac{1}{c^p}. \quad \text{[Note]}$$

$\therefore$ $$x^m/a^{p+m} = 1/c^p$$

or $$a^{p+m} = x^m c^p \quad \text{or} \quad a = (x^m c^p)^{1/(p+m)}$$

or $$a^p = (x^m c^p)^{p/(p+m)} = x^{mp/(p+m)} c^{p^2/(p+m)}.$$

Similarly $$b^p = y^{mp/(p+m)} c^{p^2/(p+m)}.$$

Substituting these values of a^p and b^p in (2), we get

$$c^{p^2/(p+m)}\{x^{mp/(p+m)} + y^{mp/(p+m)}\} = c^p$$

or $$x^{mp/(p+m)} + y^{mp/(p+m)} = c^{p-p^2/(p+m)}$$

or $$x^{mp/(p+m)} + y^{mp/(p+m)} = c^{mp/(p+m)}$$

which is the required envelope.

Q.5. Trace the curve $x = a\cos t + \frac{1}{2}a\log\tan^2\frac{t}{2}$, $y = a\sin t$.

Ans. (i) Put $-t$ for t in the given equation of the curve, we get $x = a\cos t + \frac{1}{2}a\log\tan^2\frac{t}{2}$ and $y = -a\sin t$. Therefore, every value of x there are two equal and opposite value of $y \Rightarrow$ curve is symmetric about x-axis.

Further, put $\pi - t$ for t in the given equation of the curve we get

$$x = -a\cos t + \frac{1}{2}a\log\cot^2\frac{t}{2} = -a\cos t - \frac{1}{2}a\log\tan^2\frac{t}{2}$$

and $\quad y = a\sin t$

For every value of y there are two equal and opposite value of x, so curve is symmetric about y-axis.

(ii) Differentiating the given equation w.r.t. t, we get

$$\frac{dx}{dt} = -a\sin t + \frac{1}{2}a\frac{1}{\tan^2\frac{1}{2}}\left(2\tan\frac{t}{2}\sec^2\frac{t}{2}\right).\frac{1}{2}$$

$$= -a\sin t + \frac{a}{2\sin\frac{t}{2}\cos\frac{t}{2}} = -a\sin t + \frac{a}{\sin t}$$

$$= \frac{a(1-\sin^2 t)}{\sin t} = \frac{a\cos^2 t}{\sin t}$$

and $\quad \dfrac{dy}{dt} = a\cos t$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t \,.\sin t}{a\cos^2 t} = \tan t$$

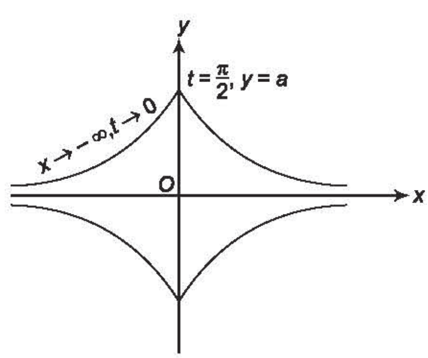
(iii) We have $y = 0$ when $\sin t = 0$, *i.e.*, $t = 0$, when $t \to 0$, $x \to -\infty$. Therefore $x \to -\infty$ when $y \to 0$ showing that the line $y = 0$ is an asymptotes of the curve.

(iv) Clearly y is maximum when $\sin t = 1$, *i.e.*, $t = \dfrac{\pi}{2}$

$$t = \frac{\pi}{2}, x = 0, y = a \text{ and } \frac{dy}{dx} = \tan\frac{\pi}{2} = \infty$$

$\Rightarrow$ Curve passes through the point $(0, a)$ and the tangent at this point is the x-axis.

(v) Clearly the numerical value of y cannot be greater than a therefore, curve does exist in the region $y > a$ and $y < -a$.



Q.6. Trace the curve $x = a(t + \sin t), y = a(1 - \cos t)$, when $-\pi \le t \le \pi$. (Cycloid)

Ans. Here $\dfrac{dx}{dt} = a(1 + \cos t)$ and $\dfrac{dy}{dt} = a \sin t$.

Therefore $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{a \sin t}{a(1 + \cos t)} = \tan \dfrac{t}{2}$.

(i) $y = 0$, when $\cos t = 1$ *i.e.*, $t = 0$.

When $t = 0, x = 0, (dy/dx) = \tan 0 = 0$.

Therefore the curve passes through the origin and the axis of x is tangent at the origin.

(ii) y is maximum when $\cos t = -1$, *i.e.*, $t = \pi$ and $-\pi$. When $t = \pi, x = a\pi, y = 2a$ and $(dy/dx) = \infty$.

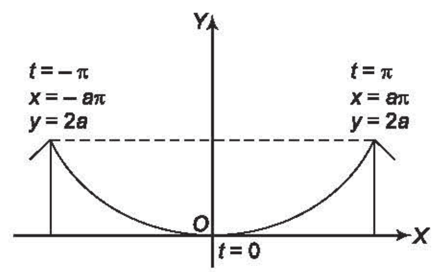
Therefore at the point $t = \pi$, whose cartesian coordinates are $(a\pi, 2a)$, the tangent is perpendicular to the x-axis. When $t = -\pi, x = -a\pi, y = 2a, (dy - dx) = -\infty$.

(iii) In this curve y cannot be negative. Therefore the curve lies entirely above the axis of x. Also no portion of the curve lies in the region $y > 2a$.

(iv) Corresponding values of x, y and (dy/dx) for different values of t are given in the following table :

t	$-\pi$	$-\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	π
x	$-a\pi$	$-a(\frac{1}{2}\pi + 1)$	0	$a(\frac{1}{2}\pi + 1)$	$a\pi$
y	$2a$	a	0	a	$2a$
dy/dx	$-\infty$	-1	0	1	∞

If we put $-t$ in place of t in the equation of the curve, we get $x = -a(t + \sin t)$, and $y = a(1 - \cos t)$. Thus for every value of y, there are two equal and opposite values of x. Therefore the curve is symmetrical about the y-axis. Hence the shape of the curve is as shown in the diagram. The portion of the cycloid included between two successive cusps is called an **arch** of the cycloid.



Q.7. Trace the curve $y^2(a + x) = x^2(a - x)$.

Ans. (i) The curve is symmetrical about x-axis.

(ii) The curve passes through the origin. The tangents at origin are $a(y^2 - x^2) = 0$ *i.e.*, $y = \pm x$. Since there are two real and distinct tangents at the origin, therefore the origin is a node on the curve.

(iii) The curve intersects the x-axis where $y = 0$

i.e., $x^2(a - x) = 0$.

Therefore the curve intersects the x-axis at $(0, 0), (a, 0)$.

The curve intersects the y-axis only at origin.

(iv) **Tangent at $(a, 0)$:** Shifting the origin to $(a, 0)$ the equation of the curve becomes

$$y^2\,(2a+x)=(x+a)^2\,\{a-(x+a)\} \quad \text{or} \quad y^2\,(2a+x)=-x\,(x^2+2ax+a^2).$$

Equating to zero the lowest degree terms, we get $x=0$ (*i.e.,* the new y-axis) as the tangent at the new origin. Thus the tangent at $(a, 0)$ is perpendicular to x-axis.

(v) Solving the equation of the curve for y, we get

$$y^2=x^2\,(a-x)/(x+a).$$

When $x=0$, $y^2=0$ and when $x=a$, $y^2=0$.

When $0<x<a$, y^2 is positive. Therefore the curve exists in this region.

When $x>a$, y^2 is negative. Therefore the curve does not exist in the region $x>a$.

When $x\to -a$, $y^2\to\infty$. Therefore $x=-a$ is an asymptote of the curve.

When $-a<x<0$, y^2 is positive. Therefore the curve exists in this region.

When $x<-a$, y^2 is negative. Therefore the curve does not exist in the region $x<-a$.

(vi) The curve has an asymptote parallel to x-axis and it is $x+a=0$. Putting $y=m$ and $x=1$ in the highest *i.e.,* third degree terms in the equation of the curve, we get $\phi_3(m)=m^2+1$. The roots of the equation $\phi_3(m)=0$ are imaginary. Therefore $x=-a$ is the only real asymptote of the curve.

(vii) For the portion of the curve lying in the first quadrant, we have

$$y=x\sqrt{\{(a-x)/(a+x)\}}=x\,\frac{(1-x/a)^{1/2}}{(1+x/a)^{1/2}}.$$

When $0<x<a$, y is less than x. Therefore the curve lies below the line $y=x$ which is tangent at the origin.

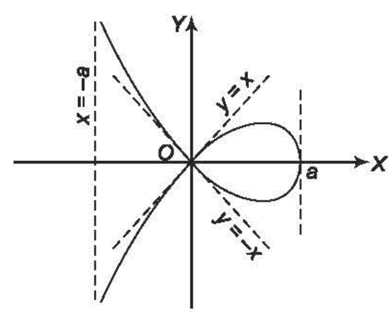
For the portion of the curve lying in the second quadrant, we have

$$y=-x\,\frac{(1-x/a)^{1/2}}{(1+x/a)^{1/2}},\; x<0.$$

When $-a<x<0$, y is greater than the numerical value of x.

Therefore the curve lies above the tangent $y=-x$.

Hence the shape of the curve is an shown in the adjoining figure.



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UNIT IV

SECTION-A VERY SHORT ANSWER TYPE QUESTIONS

Q.1. Define the sequences.

Ans. Let S be any non-empty set. A function whose domain is the set N of natural numbers and whose range is a subset of S, is called a sequence in the set S.

Q.2. Define limit point of the sequence.

Ans. A real number l is called a limit point of a sequence $< S_n >$ if every *nbd* of l contains infinite number of terms of the sequence. Thus $l \in R$ is a limit point of the sequence $< S_n >$ if for given $\varepsilon > 0, S_n \in]l-\varepsilon, l+\varepsilon[$, for infinitely many points.

Q.3. Define sub sequences.

Ans. Let $< S_n >$ be any sequence. If $(n_1, n_2, \ldots, n_k \ldots)$ be a strictly increasing sequence of positive integers *i.e.,* $i > j \Rightarrow n_i > n_j$, then the sequence $(S_{n_1}, S_{n_2}, \ldots, S_{n_k} \ldots)$ is called a sub sequence of $< S_n >$.

Q.4. Show that the sequence $\langle (-1)^n / n \rangle$ is convergent.

Ans. Let $\langle s_n \rangle = \langle (-1)^n / n \rangle$.

Here
$$\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} \frac{(-1)^{2n}}{2n} = \lim_{n\to\infty} \frac{1}{2n} = 0$$

and
$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} \frac{(-1)^{2n+1}}{2n+1} = \lim_{n\to\infty} \frac{1}{2n+1} = 0$$

which gives,
$$\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n+1} = 0 \Rightarrow \lim_{n\to\infty} s_n = 0, \ \forall n \in N.$$

Since 0 is a finite quantity. Hence, the given sequence $\langle s_n \rangle$ is a convergent sequence.

Q.5. Show that the sequence $\left\langle \frac{1}{n} \right\rangle$ converges to 0.

Ans. Let $\langle s_n \rangle = \left\langle \frac{1}{n} \right\rangle$.

Now
$$\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} \frac{1}{2n} = 0 \quad \text{and} \quad \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} \frac{1}{2n+1} = 0$$

Therefore
$$\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n+1} = 0 \Rightarrow \lim_{n\to\infty} s_n = 0, \ \forall n \in N.$$

Since 0 is a finite quantity. Hence, the sequence $\langle s_n \rangle$ is convergent and converges to 0.

Q.6. If $< t_n >$ diverges to ∞ and $s_n > t_n \ \forall n$, then $< s_n >$ diverges to ∞.

Ans. Take any given $k > 0$.

Since $< t_n >$ diverges to ∞, therefore, for $k>0$ there exists $m \in N$ such that

$t_n > k$ for all $n \geq m$

$\Rightarrow$ $s_n > k$ for all $n \geq m$. $[\because s_n > t_n \; \forall \, n \in N]$ Hence $< s_n >$ diverges to ∞.

Q.7. Show that the sequence $< \log \frac{1}{n} >$ diverges to $-\infty$.

Ans. Let $s_n = \log \frac{1}{n}$. Take any given $k<0$.

Then $s_n < k$ if $\log \frac{1}{n} < k$ *i.e.*, if $-\log n < k$

i.e., if $\log n > -k$ *i.e.*, if $n > e^{-k}$.

If we take $m \in N$ such that $m > e^{-k}$, then $s_n < k$ for all $n \geq m$. Hence $s_n \to -\infty$ as $n \to \infty$.

Q.8. Prove that the sequence $< n^p >$ where $p>0$ diverges to infinity.

Ans. Let $s_n = n^p$. Then $s_n > 0$ for all n as $n \in N$ and $p>0$.

$\therefore$ The sequence $< \frac{1}{s_n} > = < \frac{1}{n^p} >$ exists.

Since we know that $\frac{1}{n^p} \to 0$ as $n \to \infty$, $\therefore n^p \to \infty$ as $n \to \infty$. Hence $< n^p >$ diverges to ∞.

Q.9. Prove that $\lim \frac{1}{n}(1+2^{1/2}+3^{1/3}+\ldots+n^{1/n})=1$.

Ans. Let $s_n = n^{1/n}$. Then we know that $\lim n^{1/n} = 1$.

Hence, by Cauchy's first theorem on limits,

$$\lim \frac{1}{n}(s_1+s_2+\ldots+s_n)=1 \quad \text{or} \quad \lim \frac{1}{n}(1+2^{1/2}+3^{1/3}+\ldots+n^{1/n})=1.$$

Q.10. Show that $\lim \frac{1}{n}\left(1+\frac{1}{3}+\ldots+\frac{1}{2n-1}\right)=0$.

Ans. Let $s_n = \frac{1}{2n-1}$. Then $\lim s_n = \lim \frac{1}{2n-1} = 0$.

$\therefore$ By Cauchy's first theorem on limits, $\lim \frac{s_1+s_2+\ldots+s_n}{n} = 0$.

Since $s_1 = 1, s_2 = \frac{1}{3}, \ldots, s_n = \frac{1}{2n-1}$, $\quad \therefore \lim \frac{1}{n}\left\{1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right\}=0.$

Q.11. Test for convergence $\Sigma \frac{1}{(\log n)^n}$.

Ans. Here $u_n = \frac{1}{(\log n)^n}$. $\quad \therefore u_n^{1/n} = \frac{1}{\log n}$.

$\therefore \lim u_n^{1/n} = \lim \frac{1}{\log n} = 0$, which is < 1.

Hence by Cauchy's root test the given series is convergent.

Q.12. Assuming that $n^{1/n} \to 1$ as $n \to \infty$, show by applying Cauchy's *n*th root test that the series $\sum_{n=1}^{\infty} (n^{1/n} - 1)^n$ converges.

Ans. Here, $u_n = (n^{1/n} - 1)^n$. $\quad \therefore u_n^{1/n} = n^{1/n} - 1$.

$\therefore \quad \lim_{n \to \infty} u_n^{1/n} = \lim_{n \to \infty} (n^{1/n} - 1) = 0 < 1$.

Hence, by Cauchy's root test, the given series converges.

Q.13. Test the convergence of the series $\Sigma \left(1 + \frac{1}{n}\right)^{-n^2}$.

Ans. Here, we have $\quad u_n = \left(1 + \frac{1}{n}\right)^{-n^2} \Rightarrow (u_n)^{1/n} = \left(1 + \frac{1}{n}\right)^{-n}$

$$\Rightarrow \quad \lim_{n \to \infty} (u_n)^{1/n} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{-n} = \frac{1}{e} < 1.$$

Hence, by Cauchy's root test the given series Σu_n is convergent.

SECTION-B SHORT ANSWER TYPE QUESTIONS

Q.1. If $\langle s_n \rangle$ is a sequence in *R*, where $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ evaluate, $\lim_{n \to \infty} |a_{n+1} - a_n|$. Verify, is this sequence satisfy the Cauchy criterion.

Ans. Here $\quad s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \quad \Rightarrow \quad s_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \frac{1}{n+1}$

$\therefore \quad s_{n+1} - s_n = \frac{1}{n+1} \quad \Rightarrow \quad \lim_{n \to \infty} |s_{n+1} - s_n| = 0.$

Also, here we have $s_{2n} - s_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right)$

$$- \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \geq n\left(\frac{1}{2n}\right). \qquad \left(\frac{1}{n+1} > \frac{1}{2n} \text{ etc.}\right)$$

$\Rightarrow \quad |s_{2n} - s_n| > \frac{1}{2} \quad \forall\, n \in N.$

$\Rightarrow$ There exists a positive integer k such that $|s_n - s_k| \geq \frac{1}{2}$ whenever $n \geq k$.

$\Rightarrow$ Cauchy criterion in not satisfied.

Q.2. Write and prove Cauchy's second theorem on limits.

Ans. Theorem (Cauchy's second theorem on limits) : If $\langle s_n \rangle$ is a sequence of positive terms and $\lim_{n\to\infty} s_n = l$, then $\lim (s_1, s_2, \ldots, s_n)^{1/n} = l$.

Proof : Let $\langle t_n \rangle$ be a sequence, such that

$$t_n = \log s_n, \ \forall n \in \boldsymbol{N}.$$

Now $\lim s_n = l \Rightarrow \lim t_n = \lim \log s_n = \log l$

($\lim s_n = l \Leftrightarrow \lim \log s_n = \log l$ provided $s_n > 0$, $\forall n$ and $l > 0$)

Then, by Cauchy first theorem on limits, we have

$$\lim_{n\to\infty} \frac{t_1 + t_2 + \ldots + t_n}{n} = \lim t_n = \log l$$

$$\Rightarrow \lim_{n\to\infty} \frac{\log s_1 + \log s_2 + \ldots + \log s_n}{n} = \log l \quad \Rightarrow \lim_{n\to\infty} \frac{1}{n} \log (s_1 . s_2 . \ldots s_n) = \log l$$

$$\Rightarrow \lim \log (s_1 . s_2 . \ldots . s_n)^{1/n} = \log l \quad \Rightarrow \lim (s_1 . s_2 . \ldots . s_n)^{1/n} = l.$$

Q.3. Write and prove Cauchy convergence criterion.

Ans. Theorem (Cauchy Convergence Criterion) : A sequence converges if and only if it is a Cauchy sequence.

Proof : First, let $\langle s_n \rangle$ be a convergent sequence which converges to, say, l.

Since $s_n \to l$, therefore, for given $\varepsilon > 0$ there must exist $m \in \boldsymbol{N}$ such that

$$|s_n - l| < \varepsilon/2 \ \forall n \geq m.$$

In particular, $|s_m - l| < \varepsilon/2.$

Now
$$|s_n - s_m| = |(s_n - l) - (s_m - l)| \leq |s_n - l| + |s_m - l|$$
$$< \varepsilon/2 + \varepsilon/2 \text{ for all } n \geq m.$$

Thus $|s_n - s_m| < \varepsilon \ \forall n \geq m$, showing that $\langle s_n \rangle$ is a Cauchy sequence.

Conversely, let $\langle s_n \rangle$ be a Cauchy sequence. Then $\langle s_n \rangle$ is bounded. By Bolzano-Weierstrass theorem, $\langle s_n \rangle$ has a limit point, say l. We shall show that $s_n \to l$.

Let $\varepsilon > 0$ be given. Since $\langle s_n \rangle$ is a Cauchy sequence, there exists $m \in \boldsymbol{N}$ such that

$$|s_n - s_m| < \varepsilon/3 \ \forall n \geq m.$$

Since l is a limit point of $\langle s_n \rangle$, therefore every *nbd* of l contains infinite terms of the sequence $\langle s_n \rangle$. In particular the open interval $]l - \frac{1}{3}\varepsilon, l + \frac{1}{3}\varepsilon[$ contains infinite terms of $\langle s_n \rangle$.

Hence there exists a positive integer $k > m$ such that

$$l - \frac{1}{3}\varepsilon < s_k < l + \frac{1}{3}\varepsilon \quad i.e., \quad |s_k - l| < \varepsilon/3.$$

Now
$$|s_n - l| = |(s_n - s_m) + (s_m - s_k) + (s_k - l)|$$
$$\leq |s_n - s_m| + |s_m - s_k| + |s_k - l|$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \text{ for all } n \geq m.$$

Thus $|s_n - l| < \varepsilon$ for all $n \geq m$. $\therefore$ $\langle s_n \rangle$ converges to l.

Q.4. Show that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1.$

Ans. Let $\sqrt[n]{n} = 1 + h$, where $h \geq 0$

$$\Rightarrow \quad n = (1+h)^n = 1 + nh + \frac{n(n-1)}{2!}h^2 + \ldots + h^n \quad \Rightarrow \quad n > \frac{n(n-1)}{2}h^2, \forall n$$

$$\Rightarrow \quad h^2 < \frac{2}{n-1}, \quad \text{for } n \geq 2 \quad \Rightarrow \quad |h| < \sqrt{\left(\frac{2}{n-1}\right)}, \quad \text{for } n \geq 2.$$

Let $\varepsilon > 0$ (any positive number, however small) then

$$|h| < \sqrt{\left(\frac{2}{n-1}\right)} < \varepsilon \text{ provided, } \frac{2}{n-1} < \varepsilon^2 \text{ or } n > \frac{2}{\varepsilon^2} + 1.$$

If we take $m \in \boldsymbol{N}$ such that $m > \dfrac{2}{\varepsilon^2} + 1$

then $\qquad |h| < \varepsilon \ \forall\, n \geq m$

or $\qquad |\sqrt[n]{n} - 1| < \varepsilon \ \forall\, n \geq m \Rightarrow \lim\limits_{n\to\infty} \sqrt[n]{n} = 1.$

Q.5. State and prove Cesaro's theorem.

Ans. Cesaro's Theorem : If $\lim s_n = l_1$ and $\lim t_n = l_2$. Then

$$\lim \frac{s_1 t_n + s_2 t_n - 1 + \ldots + s_n t_1}{n} = l_1 l_2.$$

Proof : Let us define $s_n = l_1 + u_n$ and $|u_n| = U_n$.

Then $\lim u_n = 0$ and therefore $\lim U_n = 0$.

Now, by Cauchy's first theorem on limits, we have

$$\lim \frac{1}{n}[u_1 + u_2 + \ldots + u_n] = 0. \qquad \ldots(1)$$

Consider, $\dfrac{1}{n}[s_1 t_n + s_2 t_{n-1} + \ldots + s_n t_1]$

$$= \frac{l_1}{n}[t_1 + t_2 + \ldots + t_n] + \frac{1}{n}[u_1 t_n + u_2 t_{n-1} + \ldots + u_n t_1]. \qquad \ldots(2)$$

Since, the sequence $\langle t_n \rangle$ is convergent. Therefore, it is bounded. Hence, there must exist a positive real number k such that

$$|t_n| < k, \ \forall\, n \in \boldsymbol{N}.$$

Therefore, $\qquad \left| \dfrac{1}{n}(u_1 t_n + u_2 t_{n-1} + \ldots + u_n t_1 \right| \geq 0$

$$\frac{1}{n}[|u_1||t_n| + |u_2||t_{n-1}| + \ldots + |u_n||t_1|] \geq 0$$

$$\Rightarrow \qquad \frac{k}{n}[|u_1| + |u_2| + \ldots + |u_n|] > 0$$

$$\Rightarrow \qquad \frac{k}{n}[u_1 + u_2 + \ldots + u_n] > 0.$$

$$\Rightarrow \qquad \frac{k}{n}[u_1 + |u_2 + \ldots + u_n] \to 0 \text{ as } n \to \infty. \qquad \text{[By using]}$$

Thus $\lim \frac{1}{n}[u_1 t_n + u_2 t_{n-1} + \ldots + u_n t_1] = 0.$

Since, $\lim t_n = l_2$, therefore $\lim \frac{t_1 + t_1 + \ldots + t_n}{n} = l_2$.

Now, from (2), we have $\lim \frac{1}{n}(s_1 t_n + s_2 t_{n-1} + \ldots + s_n t_1) = l_1 l_2$.

Q.6. Show that the sequence $\langle s_n \rangle$ defined by

$$s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n} \textbf{ converges.}$$

Ans. Since, the sequence $\langle s_n \rangle$ is defined by

$$s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}$$

$$\Rightarrow \qquad s_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n+2}$$

Now $$s_{n+1} - s_n = \left(\frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right)$$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n-1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0, \ \forall n.$$

Hence, the sequence $\langle s_n \rangle$ is monotonically increasing.

Now $$|s_n| = \left|\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}\right| < \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} = n \cdot \frac{1}{n} = 1$$

i.e., $|s_n|, 1, \forall n.$

$\Rightarrow$ sequence $\langle s_n \rangle$ is bounded.

Then, by monotonic convergence criterion, the sequence $\langle s_n \rangle$ converges.

Q.7. If $r > 0$, show that $r^{1/n} = 1$.

Ans. There are following three cases :

Case I. When $r > 1$.

Let $s_n = r^{1/n} - 1$, then $s_n > 0, \ \forall n \in \boldsymbol{N}$, therefore

$$r^{1/n} = 1 + s_n$$

$$\Rightarrow \qquad r = [1 + s_n]^n = 1 + ns_n + \ldots + s_n{}^n, \geq 1 + ns_n, \forall n \in \boldsymbol{N}$$

$$\Rightarrow \qquad \frac{r-1}{n} \geq s_n, \ \forall n \in \boldsymbol{N}.$$

Hence $0 \le s_n \le \frac{r-1}{n}, \forall\, n \in \boldsymbol{N}.$

Then, by Sandwich theorem, we have $\lim s_n = 0 = \lim r^{1/n} = 1.$

Case II. When $r = 1$.

Here, $r^{1/n} = 1, \; \forall\, n \in \boldsymbol{N}. \quad \Rightarrow \quad \lim r^{1/n} = 1.$

Case III. When $0 < r < 1$, then $\frac{1}{r} > 1$.

$$\lim \left(\frac{1}{r}\right)^{1/n} = 1 \;\Rightarrow\; \lim \frac{1}{r^{1/n}} = 1 \;\Rightarrow\; \lim r^{1/n} = 1.$$

Q.8. Show that the sequence $\langle n / n+1 \rangle$, is a bounded monotonically increasing sequence and convergent too.

Ans. Let $\langle s_n \rangle = \left\langle \frac{n}{n+1} \right\rangle = (1/2, 2/3, 3/4, \ldots, n/n+1, \ldots,).$

Since, 1/2 < 2/3 < 3/4 < .

Now $s_{n+1} - s_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)} > 0$

$\Rightarrow \quad s_{n+1} - s_n < 0 \,\forall\, n$

$\Rightarrow \quad s_{n+1} > s_n \;\forall\, n \Rightarrow \langle s_n \rangle$ is monotonically increasing.

Further, $n \ge 1 > 0 \Rightarrow s_n > 0.$

Also $1 - s_n = 1 - \frac{n}{n+1} = \frac{1}{n+1} > 0 \Rightarrow s_n < 1 \;\forall\, n$

$\Rightarrow \quad 0 < s_n < 1, \;\forall\, n \Rightarrow \langle s_n \rangle$ is bounded also.

We know that a bounded monotonic sequence is always convergent. Therefore, given sequence is convergent.

Also $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right) = \lim\limits_{n\to\infty} \left[\frac{1}{1+1/n}\right] = 1.$

But l cannot be -1 since all terms of the sequence are positive. Hence $l = 2$.

Q.9. Show that the sequence $< s_n >$ defined by the relation

$$s_1 = 2, s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{(n-1)!} (n \ge 2), \textbf{ converges.}$$

Ans. We have $s_{n+1} - s_n = \frac{1}{n!} > 0$ for all n.

Hence, the sequence $< s_n >$ is monotonically increasing.

Now, we shall show that $< s_n >$ is bounded.

For $n \ge 2, n! = 1.2.3\ldots n$ contains $(n-1)$ factors each of which is greater than or equal to 2. Hence $n! \ge 2^{n-1}$ for all $n \ge 2$.

$\therefore \frac{1}{n!} \le \frac{1}{2^{n-1}}$, for all $n \ge 2$.

Thus $$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{(n-1)!}$$

$$\leq 1 + \frac{1}{1!} + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-2}} = 1 + \frac{1-\left(\frac{1}{2}\right)^{n-1}}{1-\frac{1}{2}} < 3, \text{ for all } n \geq 2.$$

Also $s_1 = 2 < 3.$

$\therefore$ $2 \leq s_n < 3$ for all $n \in N$ *i.e.*, $< s_n >$ is bounded.

Since $< s_n >$ is a bounded, monotonically increasing sequence, consequently, it converges.

Q.10. Prove that the sequence $\left\langle \frac{2n-7}{3n+2} \right\rangle$ is monotonically increasing bounded above and bounded below.

Ans. A sequence $\left\langle \frac{2n-7}{3n+2} \right\rangle$ is said to be a monotonic sequence, if it is either an increasing sequence or a decreasing sequence; if either $s_{n+1} \geq s_n$ or $s_{n+1} \leq s_n \ \forall \ n \in N$.

Since $$s_n = \frac{2n-7}{3n+2} = \frac{\frac{2}{3}(3n+2) - \frac{4}{3} - 7}{3n+2} = \frac{2}{3} - \frac{25}{3} \cdot \frac{1}{3n+2}$$

$\therefore$ $$s_{n+1} = \frac{2}{3} - \frac{25}{3} \cdot \frac{1}{2n+5}$$

$\therefore$ $$s_{n+1} - s_n = \frac{25}{3}\left[\frac{1}{3n+2} - \frac{1}{3n+5}\right] = \frac{25}{(3n+2)(3n+5)} > 0 \ \forall \ n \in N$$

$\therefore s_{n+1} > s_n \ \forall \ n \in N$ and so the sequence $\langle s_n \rangle$ is a monotonic increasing sequence.

Also from (1), we observe that $s_n < \frac{2}{3} \ \forall \ n \in N$ *i.e.*, 2/3 is an upper bound for $\langle s_n \rangle$ and so $\langle s_n \rangle$ is bounded above.

Again $\langle s_n \rangle$ is a monotonic increasing sequence and so

$$s_n \geq s_1 = \frac{2-7}{3+2} = -1 \ \forall \ n \in N.$$

$\therefore$ $s_1 = -1$ is a lower bound for $\langle s_n \rangle$ and $\langle s_n \rangle$ is bounded below.

Since the sequence $\langle s_n \rangle$ is monotonic increasing and bounded above, therefore by monotonic convergence theorem $\langle s_n \rangle$ converges to its supremum.

From (1), we observe that $\lim\limits_{n \to \infty} s_n = \frac{2}{3} - 0 = \frac{2}{3}$.

Thus the sequence $\langle s_n \rangle$ converges to $\frac{2}{3}$ and so $\sup(s_n) = \frac{2}{3}$.

Also $\inf \langle s_n \rangle = s_1 = -1.$

Q.11. Test the convergence or divergence of the series $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\ldots$.

Ans. Leaving the first term, we get

$$u_n=\frac{n^n}{(n+1)^{n+1}}=\frac{1}{n\left(1+\frac{1}{n}\right)^{n+1}}=\frac{1}{n}\left[1+\frac{1}{n}\right]^{-[n+1]}=\frac{1}{n}\left[1-\frac{(n+1)}{n}+\ldots\right]=\frac{1}{n}-\left(1+\frac{1}{n}\right)\frac{1}{n}+\ldots$$

Let $\Sigma v_n=\Sigma\frac{1}{n}$, where $v_n=\frac{1}{n}$, be the auxiliary series.

Then $$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left[\frac{1}{n}\cdot\frac{1}{(1+1/n)^{n+1}}\right]\Big/\frac{1}{n}$$

$$=\lim_{n\to\infty}\left[\frac{1}{(1+1/n)^n}\Big/\left(1+\frac{1}{n}\right)\right]=\frac{1}{e}, \text{ which is finite and non-zero.}$$

Now, since the series $\Sigma v_n=\Sigma\frac{1}{n}$ is divergent, therefore by comparison test the given series is also divergent.

Q.12. Test the convergence of the following series

(i) $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots$ (ii) $\frac{1}{1+\sqrt{2}}+\frac{2}{1+2\sqrt{3}}+\frac{3}{1+3\sqrt{4}}+\ldots$

Ans. (i) Omitting the first term, if the given series is denoted by Σu_n, then

$$\Sigma u_n=\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots=\Sigma\frac{n^n}{(n+1)^{n+1}}.$$

Here, $u_n=\frac{n^n}{(n+1)^{n+1}}$. Take $v_n=\frac{n^n}{n^{n+1}}=\frac{1}{n}$.

Now $$\lim\frac{u_n}{v_n}=\lim\left\{\frac{n^n}{(n+1)^{n+1}}\cdot n\right\}$$

$$=\lim\left\{\frac{1}{(1+1/n)^n\cdot(1+1/n)}\right\}=\frac{1}{e}, \qquad \left[\because \lim\left(1+\frac{1}{n}\right)^n=e\right]$$

which is finite and non-zero.

But the auxiliary series $\Sigma v_n=\Sigma(1/n)$ is divergent as here $p=1$. Hence by comparison test the given series is divergent.

(ii) Here, $$u_n=\frac{n}{1+n\sqrt{(n+1)}}$$

Take $$v_n=\frac{n}{n\sqrt{n}}=\frac{1}{n^{1/2}}.$$

Now $$\lim \frac{u_n}{v_n} = \lim \left\{ \frac{n}{1 + n\sqrt{(n+1)}} . n^{1/2} \right\}$$

$$= \lim \left\{ \frac{1}{1/n^{3/2} + \sqrt{(1 + 1/n)}} \right\} = 1, \text{ which is finite and non-zero.}$$

Since the auxiliary series $\Sigma\, v_n = \Sigma\, (1/n^{1/2})$ is divergent as here $p = 1/2 < 1$, therefore, by comparison test the given series is divergent.

Q.13. State and prove Cauchy's integral test.

Ans. Cauchy's Integral Test : Let $f(x)$ is a non-negative monotonically decreasing integrable function on $[1, \infty[$ then the series $\sum_{n=1}^{\infty} f(n)$ and the improper integral $\int_1^{\infty} f(x)\, dx$ converge or diverge together.

Proof : Let $f(x)$ is a monotonically decreasing on $[1, \infty]$.

Then we have $f(n) \geq f(x) \geq f(n+1)$, where $n \leq x \leq n+1$.

Also $f(x)$ in non-negative and integrable, we have

$$\int_n^{n+1} f(n)\, dx \geq \int_n^{n+1} f(x)\, dx \geq \int_n^{n+1} f(n+1)\, dx$$

or $$f(n) \geq \int_n^{n+1} f(x)\, dx \geq f(n+1). \qquad ...(1)$$

Now, putting $n = 1, 2, \ldots (n-1)$ in (1) and adding all these, we get

$$f(1) + f(2) + \ldots + f(n-1) \geq \int_1^2 f(x)\, dx + \int_2^3 f(x)\, dx + \ldots$$

$$+ \int_{n-1}^{n} f(x\, dx \geq f(2) + f(3) + \ldots f(n). \qquad ...(2)$$

Let us suppose $S_n = f(1) + f(2) + \ldots + f(n)$

and $I_n = \int_1^n f(x)\, dx.$

Then (2) can be written as

$$S_n - f(x) \geq I_n \geq S_n - f(1)$$

or $$f(n) \leq S_n - I_n \leq f(1). \qquad ...(3)$$

Let $u_n = S_n - I_n \ \forall\, n \in N.$

Then $$u_{n+1} - u_n = (S_{n+1} - I_{n+1}) - (S_n - I_n)$$

$$= (S_{n+1} - S_n) - (I_{n+1} - I_n) = f(n+1) - \int_n^{n+1} f(x)\, dx$$

$$\leq 0 \qquad \text{[using (1)]}$$

Hence, we have $\langle u_n \rangle$ is monotonically decreasing sequence.

Now, from (3) $u_n \geq f(n) \geq 0, \forall\, n \in \boldsymbol{N}$. Therefore sequence $\langle u_n \rangle$ is bounded below. Hence $\langle u_n \rangle$ is a convergent sequence and it has a finite limit.

Now, since $S_n = u_n + I_n$, the sequence $\langle S_n \rangle$ and $\langle I_n \rangle$ converge or diverge together. Hence, the series $\Sigma f(n)$ and the integral $\int_1^{\infty} f(x)\, dx$ converge or diverge together.

Q.14. Show that Cauchy's integral test that the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges $p>1$ and diverges if $0<p\leq 1$.

Ans. Let us suppose $$f(x)=\frac{1}{x(\log x)^p}, p>0$$

and $x \in [2,\infty[$; then obviously $f(x)$ is monotonically decreasing on $[2,\infty[$ and positive value

Let $$I_n = \int_2^n \frac{dx}{x(\log x)^p}$$

Then $$I_n = \left[\frac{(\log x)^{1-p}}{1-p}\right]_2^n, p\neq 1$$

$$=\frac{1}{(1-p)}[(\log n)^{1-p} - (\log 2)^{1-p}], p\neq 1$$

and $$I_n = [\log\log x]_2^n, p=1$$

$$=[\log\log n - \log\log 2], p=1$$

Therefore, we have $\lim\limits_{n\to\infty} I_n = \lim\limits_{n\to\infty} \int_2^n f(x)\,dx = \infty$, if $p<1$

and $$\lim_{n\to\infty} I_n = -\frac{1}{(1-p)}(\log 2)^{1-p}, \text{ if } p>1.$$

Thus the integral $\int_2^{\infty} f(x)\,dx$ converges if $p>1$ and diverges if $0<p\leq 1$.

Hence, by Cauchy's integral test, the series

$$\sum_{n=2}^{\infty} f(x) = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges if $p>1$ and diverges if $0<p\leq 1$.

Q.15. Test the converges of the series

$$\frac{1}{2}+\left(\frac{2}{3}\right)x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\ldots\infty, x>0.$$

Ans. Omitting the first term of the series (because it will not affect the convergence or emergence of the series), we have

$$u_n = \left(\frac{n+1}{n+2}\right)^n . x^n.$$

Therefore, $$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left[\frac{\left(1+\frac{1}{n}\right)x}{1+\left(\frac{2}{n}\right)}\right] = x$$

Therefore, by Cauchy's root test, the given series Σu_n converges if $x<1$, divergent if $x>1$. For $x=1$, test fails

$$\therefore \qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \frac{e}{e^2} = \frac{1}{e} > 0.$$

$\therefore$ The series Σu_n diverges if $x=1$.

Hence, the given series is convergent if $x<1$ and divergent if $x\ge 1$.

Q.16. Test for convergence the following series :

(i) $1+\dfrac{2^p}{2!}+\dfrac{3^p}{3!}+\dfrac{4^p}{4!}+\ldots$ **(ii)** $\dfrac{1}{1+2}+\dfrac{2}{1+2^2}+\dfrac{3}{1+2^3}+\ldots$

Ans. (i) Here $u_n = \dfrac{n^p}{n!}$ $\quad\therefore\quad u_{n+1} = \dfrac{(n+1)^p}{(n+1)!}$.

Now $$\frac{u_n}{u_{n+1}} = \frac{n^p}{n!}\cdot\frac{(n+1)!}{(n+1)^p} = \frac{(n+1)\,n^p}{(n+1)^p} = \frac{n+1}{(1+1/n)^p}.$$

$$\therefore \qquad \lim \frac{u_n}{u_{n+1}} = \lim \frac{n+1}{(1+1/n)^p} = \infty, \text{ which is} >1 \text{ for all values of } p.$$

Hence by ratio test the series $\Sigma\, u_n$ is convergent.

(ii) Here $u_n = \dfrac{n}{1+2^n}$ $\quad\therefore\quad u_{n+1} = \dfrac{n+1}{1+2^{n+1}}$.

Now $$\frac{u_n}{u_{n+1}} = \frac{n}{1+2^n}\cdot\frac{1+2^{n+1}}{n+1} = \frac{n.2^{n+1}\,(1+1/2^{n+1})}{2^n\,(1+1/2^n).n\,(1+1/n)} = \frac{2\,(1+1/2^{n+1})}{(1+1/2^n)\,(1+1/n)}.$$

$$\therefore \qquad \lim \frac{u_n}{u_{n+1}} = 2.\frac{(1+0)}{(1+0)\,(1+0)} = 2, \text{ which is} >1.$$

Therefore, by ratio test, the given series converges.

Q.17. Test for convergence the following series

$$1+a+\frac{a(a+1)}{1.2}+\frac{a(a+1)(a+2)}{1.2.3}+\ldots$$

Ans. Leaving the first term, we have

$$u_n = \frac{a\,(a+1)\,(a+2)\ldots(a+n-1)}{1.2.3\ldots n},$$

and then $$u_{n+1} = \frac{a\,(a+1)\,(a+2)\ldots(a+n-1)\,(a+n)}{1.2.3\ldots n\,(n+1)}.$$

Now $$\frac{u_n}{u_{n+1}} = \frac{n+1}{a+n} = \frac{1+1/n}{a/n+1}.$$

$\therefore\ \lim \dfrac{u_n}{u_{n+1}} = 1$, so that the ratio test fails.

Now we apply Raabe's test. We have

$$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim n\left(\frac{n+1}{a+n}-1\right)=\lim \frac{n(1-a)}{a+n}=\lim \frac{(1-a)}{1+a/n}=1-a.$$

Hence by Raabe's test, the given series is convergent if $1-a>1$ *i.e.,* if $a<0$, divergent if $1-a<1$ *i.e.,* if $a>0$ and the test fails if $1-a=1$ *i.e.,* if $a=0$.

In case $a=0$, the given series becomes $1+0+0+0+\ldots$

The sum of n terms of this series is always 1. Therefore the series is convergent if $a=0$.

Thus the given series is convergent if $a \le 0$ and divergent if $a>0$.

Q.18. Test the convergence of the series $1+\dfrac{1}{2}x+\dfrac{2!}{3^2}x^2+\dfrac{3!}{4^3}x^3+\ldots$

Ans. Here, we have $u_n = \dfrac{(n-1)!}{n^{n-1}}x^{n-1} \Rightarrow u_{n+1} = \dfrac{n!}{(n+1)^n}x^n$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{(n+1)^n\,(n-1)!\,x^{n-1}}{n!\,x^n\,.\,n^{n-1}} = \lim_{n\to\infty}\left[1+\frac{1}{n}\right]^n.\frac{1}{x}=\frac{e}{x}.$$

Hence, the given series is convergent if $\dfrac{e}{x}>1$, *i.e.,* if $x<e$, divergent if $x>e$ and the test fails if $x=e$. In this case

$$\lim_{n\to\infty}\left[n\log\frac{u_n}{u_{n+1}}\right]=\lim_{n\to\infty}\left[n\log\frac{\left(1+\frac{1}{n}\right)^n}{e}\right]=\lim_{n\to\infty}\left[n^2\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+\ldots\right)-n\right]$$

$$=\lim_{n\to\infty}\left[-\frac{1}{2}+\frac{1}{3n}-\ldots\right]=-\frac{1}{2}<1.$$

Hence, by log test the series Σu_n is divergent if $x=e$.

Thus the given series Σu_n is convergent if $x<e$ and divergent if $x \ge e$.

Q.19. State and prove Leibnitz test.

Ans. **Leibnitz Test :** If the alternative series $u_1-u_2+u_3-\ldots\ (u_n>0, \forall n\in N)$ is such that

(i) $u_{n+1} \le u_n\ \forall\, n \in N$ (ii) $\lim\limits_{n\to\infty} u_n = 0$.

Then the series converges.

Proof : Let $s_n = u_1-u_2+u_3-\ldots+(-1)^{n-1}u_n$ so that $<s_n>$ is a sequence of partial sums of the given series.

Now for all n, $s_{2n+2}-s_{2n}=u_{2n+1}-u_{2n+2}\ge 0$ [By (1)]

which gives that s_{2n} is a monotonically increasing sequence.

Further, $s_{2n} = u_1 - u_2 + u_3 - \dots u_{2n-1} - u_{2n}$

$$= u_1 - (u_2 + u_3) - (u_4 - u_5) - \dots - u_{2n}$$
$$= u_1 - [(u_2 - u_{3)} + \dots + u_{2n}]$$
$$= u_1 - \text{some positive number}$$
$$\leq u_1.$$

Therefore, the monotonically increasing sequence $< s_{2n} >$ is bounded above and consequently it is convergent.

Let $\lim\limits_{n \to \infty} s_{2n} = s.$

Now $s_{2n+1} = s_{2n} + u_{2n+1}$

$$\lim_{n \to \infty} s_{2n+1} = \lim_{n \to \infty} s_{2n} + \lim_{n \to \infty} u_{2n+1} = s + 0 \qquad \left[\because \lim_{n \to \infty} u_n = 0\right]$$
$$= s.$$

Thus, the subsequences $< s_{2n} >$ and $< s_{2n+1} >$ both converge to the same limits. Now we shall show that the sequence $< s_n >$ also converges to S.

Let $e > 0$ be given. Since, the sequences s_{2n} and s_{2n+1} both converges to S, there exist positive integers m_1, m_2 such that

$$|s_{2n} - s| < \varepsilon \; \forall \; n \geq m_1,$$

and $|s_{2n+1} - s| < \varepsilon \; \forall \; n \geq m_2.$

Let $m = \max(m_1, m_2).$

Then $|s_n - s| < \varepsilon \; \forall \; n \geq m$

which gives that the sequence $< s_n >$ converges to s.

Hence, the given series $\Sigma (-1)^{n-1} u_n$ converges.

Q.20. Test the convergence of the series

$$\frac{1}{x} - \frac{1}{x+a} + \frac{1}{x+2a} \dots, x > 0, a > 0.$$

Ans. Since, the given series is an alternating series.

$\therefore$ The n^{th} term

$$t_n = (-1)^{n-1} u_n, \text{ where } u_n = \frac{1}{x + (n-1)a} > 0.$$

Now $u_{n+1} - u_n = \dfrac{1}{x + na} - \dfrac{1}{x + (n-1)a}$

$$= \frac{[x + (n-1)a] - [x + na]}{[x + na][x + (n-1)a]} = \frac{-a}{[x + ma][x + (n-1)a]} > 0$$

$\therefore$ $u_{n+1} < u_n.$

Also, $\lim\limits_{n \to \infty} u_n = \lim\limits_{n \to \infty} \dfrac{1}{x + (n-1)a} = 0.$

Hence, by Leibnitz test, the given series is convergent.

Q.21. Show that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$ converges.

Ans. The given series is an alternating series.

$$u_1-u_2+u_3-u_4+\dots+(-1)^{n-1}u_n+\dots, (u_n>0 \text{ for all } n).$$

Here $u_n=1/n>0$ for all n.

We have $$u_{n+1}-u_n=\frac{1}{n+1}-\frac{1}{n}=\frac{n-n-1}{n(n+1)}=\frac{-1}{n(n+1)}<0 \text{ for all } n.$$

Thus $u_{n+1}<u_n$ for all n *i.e.,* each term is numerically less than the preceding term.

Also $$\lim u_n=\lim\frac{1}{n}=0.$$

Hence by Leibnitz's test for alternating series, the given series is convergent.

Q.22. Show that the series $\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\dots$ is conditionally convergent.

Ans. The given series is an alternating series.

$\therefore$ The n^{th} term $$t_n=(-1)^{n-1}u_n \text{ where } u_n=\frac{1}{\sqrt{n}}>0.$$

Now $$u_{n+1}-u_n=\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n}}=\frac{\sqrt{n}-\sqrt{n+1}}{\sqrt{n}\sqrt{n+1}}<0.$$

$\therefore$ $$u_{n+1}<u_n.$$

Also $$\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0.$$

$\therefore$ By Leibnitz test the given series is convergent.

But the series $\Sigma\left|\frac{(-1)^{n-1}}{\sqrt{n}}\right|=\Sigma\frac{1}{\sqrt{n}}$ is divergent $\left(\because p=\frac{1}{2}<1\right)$

Hence, the given series is conditionally convergent.

SECTION-C LONG ANSWER TYPE QUESTIONS

Q.1. Write and prove Cauchy's first theorem on limits.

Ans. Theorem (Cauchy's first theorem on limits) :

If $\lim_{n\to\infty}s_n=l$, then $\lim_{n\to\infty}\frac{s_1+s_2+\dots+s_n}{n}=l.$

Proof : Define a sequence $<t_n>$ such that

$$s_n=l+t_n\ \forall n\in\mathbf{N}.$$

$\therefore$ $$\lim t_n=0$$

and $$\frac{s_1+s_2+\dots+s_n}{n}=l+\frac{t_1+t_2+\dots+t_n}{n}. \quad \dots(1)$$

In order to prove the theorem we wish to show that

$$\lim_{n\to\infty} \frac{t_1 + t_2 + \ldots t_n}{n} = 0.$$

Let $\varepsilon > 0$ be given. Since $\lim t_n = 0$, therefore, there exists a positive integer m, such that

$$|t_n - 0| = |t_n| < \varepsilon/2 \ \forall\, n \ge m. \qquad \ldots(2)$$

Also, since every convergent sequence is bounded, hence there exists a real number $k > 0$ such that

$$|t_n| \le k \ \forall\, n \in \mathbf{N}. \qquad \ldots(3)$$

Now for all $n \ge m$, we have

$$\left|\frac{t_1 + t_2 + \ldots + t_n}{n}\right| = \left|\frac{t_1 + t_2 + \ldots + t_m}{n} + \frac{t_{m+1} + t_{m+2} + \ldots + t_n}{n}\right|$$

$$\le \frac{|t_1| + |t_2| + \ldots + |t_m|}{n} + \frac{|t_{m+1}| + |t_{m+2}| + \ldots + |t_n|}{n}$$

$$\le \frac{mk}{n} + \frac{n-m}{n} \cdot \frac{\varepsilon}{2} \qquad \text{[From (2) and (3)]}$$

$$< \frac{mk}{n} + \frac{\varepsilon}{2}. \qquad \ldots(4)$$

$$\left[\because\ 0 \le \frac{n-m}{n} < 1\right]$$

If m is fixed, then $\frac{mk}{n} < \frac{1}{2}\varepsilon$ if $n > \frac{2mk}{\varepsilon}$.

Let us choose a positive integer $p > \frac{2mk}{\varepsilon}$. Then

$$\frac{mk}{n} < \frac{1}{2}\varepsilon \text{ for } n \ge p. \qquad \ldots(5)$$

Let $M = \max.\{m, p\}$. From (4) and (5), we have

$$\left|\frac{t_1 + t_2 + \ldots + t_n}{n}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \forall\, n \ge M.$$

Thus $\lim \frac{t_1 + t_2 + \ldots + t_n}{n} = 0$ and consequently (1) gives

$$\lim \frac{s_1 + s_2 + \ldots + s_n}{n} = l.$$

Q.2. Show by applying Cauchy's convergent criterion that the sequence $\langle s_n \rangle$ given by $s_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}$ diverges.

Ans. Here, we have

$$s_{n+1} = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} + \frac{1}{2(n+1)-1}$$

$$= 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} + \frac{1}{2n+1}$$

$$s_{n+1}-s_n=\left[1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}+\frac{1}{2(n+1)-1}\right]-\left[1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right]$$

$$=\frac{1}{2n+1}>0,\ \forall n\in \boldsymbol{N}.$$

$$s_{n+1}>s_n,\ \forall n\in \boldsymbol{N}.$$

$\Rightarrow$ The sequence $\langle s_n\rangle$ is increasing sequence.

Also, we have $s_{2n}=1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}+\frac{1}{2n+1}+\ldots+\frac{1}{4n-1}$

$$s_{2n}-s_n=\left[1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}+\frac{1}{2n+1}+\ldots+\frac{1}{4n-1}\right]-\left[1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right]$$

$$=\frac{1}{2n+1}+\frac{1}{2n+3}+\ldots+\frac{1}{4n-1}$$

$\Rightarrow$ $s_{2n}-s_n>n\left(\frac{1}{4n}\right)$ $\quad\left(\frac{1}{2n+1}>\frac{1}{4n}\text{ etc. and there are } n \text{ terms}\right)$

$\Rightarrow$ $|s_{2n}-s_n|>\frac{1}{4},\ \forall n\in \boldsymbol{N}$

$\Rightarrow$ there exists a positive integer k such that $|s_n-s_k|>\frac{1}{4}$ whenever $n\geq k$

$\Rightarrow$ Cauchy criterion is not satisfied.

$\Rightarrow$ The sequence $\langle s_n\rangle$ can not converge.

$\Rightarrow$ The sequence $\langle s_n\rangle$ diverges to $+\infty$.

Q.3. Write and prove Cauchy's general principle of convergence.

Ans. Theorem (Cauchy's general principle of convergence) : A sequence is convergent if and only if it is a Cauchy sequence.

Proof : Let us first suppose $\langle s_n\rangle$ be a convergent sequence. Let, this sequence converges to

$\therefore$ for a given $\varepsilon>0$ these exists a positive integer m such that

$$|s_n-l|<\varepsilon/2,\ \forall n\geq m. \qquad ...(1)$$

In particular, for $n=m$

$$|s_m-l|<\varepsilon/2. \qquad ...(2)$$

Now, consider $|s_n-s_m|=|s_n-l+l-s_m|\leq|s_n-l|+|s_m-l|$

$$<\varepsilon/2+\varepsilon/2,\ \forall n\geq m$$

$$=\varepsilon,\ \forall n\geq m$$

i.e., $|s_n-s_m|<\varepsilon,\ \forall n\geq m$

$\Rightarrow$ $\langle s_n\rangle$ is a Cauchy sequence.

Conversely, Set $\langle s_n\rangle$ be a Cauchy sequence.

$\Rightarrow$ $\langle s_n\rangle$ is a bounded sequence. [By Theorem 1]

$\Rightarrow$ By Bolzano-Wierstress theorem $\langle s_n \rangle$ has at least one limit point, say l. We shall show that sequence (s_n) converges to l.

Let $\varepsilon > 0$ be given.

Since, $\langle s_n \rangle$ is a Cauchy sequence.

$\therefore$ $\exists$ a positive integer m such that

$$|s_n - s_m| < \varepsilon/3, \ \forall\, n \geq m.$$

Since, l is the limit point of $\langle s_n \rangle$.

$\therefore$ for above choice of ε and m, $\exists$ a positive integer $k > m$ such that

$$|s_k - l| < \varepsilon/3.$$

Since, $k > m$, therefore from (3)

$$|s_k - s_m| < \varepsilon/3.$$

Now, consider
$$\begin{aligned}|s_n - l| &= |s_n - s_m + s_m - s_k + s_k - l| \\ &\leq |s_n - s_m| + |s_m - s_k| + |s_k - l| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon\end{aligned}$$

i.e.,
$$|s_n - l| < \varepsilon, \ \forall\, n \geq m.$$

Hence, $\langle s_n \rangle$ is convergent.

Q.4. Prove that $\displaystyle \lim_{n\to\infty}\left[\frac{(n+1)(n+2)(n+3)\ldots(n+n)}{n^n}\right]^{1/n} = \frac{4}{e}$.

Ans. Let
$$s_n = \frac{(n+1)(n+2)\ldots(n+n)}{n^n} = \frac{(2n)!}{n^n\,(n!)}$$

Then
$$s_{n+1} = \frac{(2n+2)!}{(n+1)^{n+1}\,(n+1)!}$$

Therefore,
$$\frac{s_{n+1}}{s_n} = \frac{(2n+2)!\,n^n\,(n!)}{(n+1)^{n+1}\,(n+1)\,(2n)!} = \frac{(2n+2)(2n+1)\,n^n}{(n+1)^{n+2}}$$

$$= \frac{(2n+2)(2n+1)\,n^n}{(n+1)^{n+2}} = \frac{2(2n+1)\,n^n}{(n+1)^{n+1}}$$

$$= \frac{2\times 2n\left[1+\frac{1}{2n}\right]n^n}{(n+1)(n+1)^n} = \frac{4n\left[1+\frac{1}{2n}\right]n^n}{n\left[1+\frac{1}{n}\right](n+1)^n}$$

$$= \frac{4\left[1+\frac{1}{2n}\right]}{\left(1+\frac{1}{n}\right)}\cdot\left[\frac{n}{n+1}\right]^n = \frac{4\left[1+\frac{1}{2n}\right]}{\left[1+\frac{1}{n}\right]}\cdot\frac{1}{\left[1+\frac{1}{n}\right]^n}$$

Now, taking $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \lim_{n\to\infty} \left[\frac{4\left[1+\frac{1}{2n}\right]}{1+\frac{1}{n}} \cdot \frac{1}{\left[1+\frac{1}{n}\right]^n} \right] = \frac{4}{e}.$$

Now by Cauchy's second theorem on limits, we have

$$\lim_{n\to\infty} (s_n)^{1/n} = \lim_{n\to\infty} \left(\frac{s_{n+1}}{s_n} \right) = \frac{4}{e} = \lim_{n\to\infty} \left[\frac{(n+1)(n+2)\dots(n+n)}{n^n} \right]^{1/n} = \frac{4}{e}.$$

Q.5. Show that the sequence $< s_n >$ defined by $s_1 = 1$, $s_{n+1} = \dfrac{4+3s_n}{3+2s_n}$, $n \in N$ is convergent and find its limit.

Ans. We observe that all the terms of the given sequence are positive.

First by mathematical induction we shall show that

$$s_{n+1} > s_n \ \forall\, n \in N.$$

We have $\quad s_1 = 1, s_2 = \dfrac{4+3s_1}{3+2s_1} = \dfrac{4+3\cdot 1}{3+2\cdot 1} = \dfrac{7}{5}. \qquad \therefore\ s_2 > s_1.$

Now assume as our induction hypothesis that for some positive integer n,

$$s_{n+1} > s_n. \qquad \text{...(1)}$$

Then
$$s_{n+2} - s_{n+1} = \frac{4+3s_{n+1}}{3+2s_{n+1}} - \frac{4+3s_n}{3+2s_n}$$
$$= \frac{(4+3s_{n+1})(3+2s_n) - (4+3s_n)(3+2s_{n+1})}{(3+2s_{n+1})(3+2s_n)}$$
$$= \frac{s_{n+1} - s_n}{(3+2s_{n+1})(3+2s_n)} > 0, \text{ by (1).}$$

$\therefore \qquad s_{n+2} > s_{n+1}.$

Thus $s_2 > s_1$ and if $s_{n+1} > s_n$, then we have also $s_{n+2} > s_{n+1}$.

$\therefore$ by mathematical induction $s_{n+1} > s_n, \forall\, n \in N$.

Thus the sequence $< s_n >$ is monotonic increasing.

Now we shall show that the sequence $< s_n >$ is also bounded above.

We have $\quad s_{n+1} = \dfrac{3s_n + 4}{2s_n + 3} = \dfrac{\frac{3}{2}(2s_n+3) - \frac{1}{2}}{2s_n + 3} = \dfrac{3}{2} - \dfrac{1}{2(2s_n+3)},$

showing that $s_{n+1} < \dfrac{3}{2}, \forall\, n \in N$. $\qquad$ Also, $\quad s_1 = 1 < \dfrac{3}{2}$.

Thus $s_n < \dfrac{3}{2}, \forall\, n \in N$. Therefore the sequence $< s_n >$ is bounded above by $\dfrac{3}{2}$.

Since the sequence $< s_n >$ is monotonic increasing and bounded above, therefore by monotone convergence theorem it converges to its supremum.

Let $\lim s_n = l$. Then $\lim s_{n+1} = l$.

Now, $$s_{n+1} = \frac{4+3s_n}{3+2s_n} \Rightarrow \lim s_{n+1} = \frac{4+3\lim s_n}{3+2\lim s_n}$$

$$\Rightarrow \quad l = \frac{4+3l}{3+2l} \quad \Rightarrow \quad l^2 = 2 \quad \Rightarrow \quad l = \pm\sqrt{2}.$$

Since all terms of the sequence are positive so l cannot be negative. Hence $l = \sqrt{2}$. We have $\inf < s_n > = s_1 = 1$ and $\sup < s_n > = \lim s_n = \sqrt{2}$.

Q.6. Write and prove Cauchy's root test.

Ans. Cauchy's Root Test : Theorem : Let Σu_n be a series of positive terms such that $\lim u_n^{1/n} = l$. Then

(i) Σu_n converges, if $l < 1$; (ii) Σu_n diverges, if $l < 1$;

(iii) the test fails and the series may either converges or diverge, if $l = 1$.

(Here $u_n^{1/n}$ stands for positive nth root of u_n).

Proof : Since $u_n > 0$, for all n, and $(u_n)^{1/n}$ stands for positive nth root of u_n, $\lim u_n^{1/n} = l \geq 0$. Since $\lim u_n^{1/n} = l$, therefore for $\varepsilon > 0$ there exists a positive integer m, such that

$$|u_n^{1/n} - l| < \varepsilon, \text{ for all } n > m,$$

i.e., $$l - \varepsilon < u_n^{1/n} < l + \varepsilon, \text{ for all } n > m,$$

i.e., $$(l-\varepsilon)^n < u_n < (l+\varepsilon)^n, \text{ for all } n > m. \quad ...(1)$$

(i) Let $l < 1$.

Choose $\varepsilon > 0$, such that $r = l + \varepsilon < 1$.

Then $0 \leq l < r < 1$.

From (1) we get $u_n < (l+\varepsilon)^n$ for all $n > m$ *i.e.,* $u_n < r^n$ for all $n > m$.

Since Σr^n is a geometric series with common ratio r less than unity, Σr^n is convergent. Therefore, by comparison test, Σu_n is convergent.

(ii) Let $l > 1$.

Choose $\varepsilon > 0$, such that $r = l - \varepsilon > 1$.

From (1), we get $(l-\varepsilon)^n < u_n$ for all $n > m$ *i.e.,* $u_n > r^n$ for all $n > m$.

Since Σr^n is a geometric series with common ratio greater than unity, Σr^n is divergent. Therefore, by comparison test, Σu_n is divergent.

(iii) Let $l = 1$.

Consider the series Σu_n, where $u_n = 1/n$.

Then $$u_n^{1/n} = \left(\frac{1}{n}\right)^{1/n}, \text{ so that } \lim u_n^{1/n} = 1. \qquad [\text{Note that } \lim n^{1/n} = 1].$$

Since $\Sigma (1/n)$ diverges, hence, we observe that if

$$\lim u_n^{1/n} = 1, \text{ the series } \Sigma u_n \text{ may diverge.}$$

Now, consider the series $\Sigma\, u_n$, where $u_n = 1/n^2$.

In this case also, $\lim u_n^{1/n} = 1$.

Since $\Sigma\,(1/n^2)$ converges, hence, we observe that if $\lim u_n^{1/n} = 1$, the series $\Sigma\, u_n$ may converge.

Thus the above two examples show that Cauchy's root test fails to decide the nature of the series when $l = 1$.

Q.7. State and prove D'ALEMBERT Ratio Test.

Ans. D'ALEMBERT Ratio Test : If Σu_n be a series of positive terms such that

(a) $\lim\limits_{n \to \infty} \dfrac{u_n}{u_{n+1}} = l.$

Then, if

(i) $l > 1$, the series converges,

(ii) $l < 1$, the series diverges,

(iii) $l = 1$, the series may converge of diverge and therefore the test fails.

(b) $\dfrac{u_n}{u_{n+1}} \to \infty$ **as** $n \to \infty$. Then Σu_n converges.

Proof : (a) Case (i) When $l > 1$, Let $\varepsilon > 0$ be a positive number such that $l - \varepsilon > 1$.

Now since $\lim\limits_{n \to \infty} \dfrac{u_n}{u_{n+1}} = l$, therefore, $\exists$ a positive integer m such that

$$l - \varepsilon < \frac{u_n}{u_{n+1}} < l + \varepsilon, \text{ wherever } n > m.$$

Now, putting $n = m+1, m+2, \ldots, p-1$, in succession in the above inequality, we get

$$l - \varepsilon, < \frac{u_{m+1}}{u_{m+2}} < l + \varepsilon$$

$$l - \varepsilon, < \frac{u_{m+2}}{u_{m+3}} < l + \varepsilon$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$l - \varepsilon < \frac{u_{p-1}}{u_p} < l + \varepsilon.$$

Multiplying the corresponding sides of the first part of the above inequalities, we get

$$(l - \varepsilon)^{p-1-m} < \frac{u_{m+1}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+3}} \cdots \frac{u_{p-1}}{u_p}$$

$$\Rightarrow \qquad (l - \varepsilon)^{p-1-m} < \frac{u_{m+1}}{u_p}$$

$$\Rightarrow \qquad u_p < u_{m+1}\,(l - \varepsilon)^{m+1} \cdot (l - \varepsilon)^{-p}$$

$$\Rightarrow \qquad u_p < k\,(l-\varepsilon)^{-p}, \forall\, p \geq m+2 \text{ and } k = u_{m+1}\,(l-\varepsilon)^{m+1}.$$

Since, the series $\Sigma\,(l-\varepsilon)^{-p}$ converges (being a geometric series with common ratio $(l-\varepsilon)^{-1}$, which is certainly less than unity), then by comparison test it follows that Σu_n converges.

Case (ii) When $l<1$, let $\varepsilon>0$ be a positive number such that $l+\varepsilon<1$.

Now since $\lim\limits_{n\to\infty}\dfrac{u_n}{u_{n+1}}=l$, therefore, $\exists$ a positive integer m such that

$$l-\varepsilon<\frac{u_n}{u_{n+1}}<l+\varepsilon, \forall\, n>m.$$

Putting $n=m+1, m+2, \ldots, p-1$, in succession in the second part of the above inequality, we get

$$\frac{u_{m+1}}{u_{m+2}}<l+\varepsilon,$$

$$\frac{u_{m+2}}{u_{m+3}}<l+\varepsilon$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\frac{u_{p-1}}{u_p}<l+\varepsilon$$

Multiplying the corresponding sides of the above inequalities, we have

$$\frac{u_{m+1}}{u_p}<(l+\varepsilon)^{p-1-m}$$

$$\Rightarrow \qquad u_p > u_{m+1}\,(l+\varepsilon)^{m+1}\,(l+\varepsilon)^{-p}$$

$$\Rightarrow \qquad u_p > A\,(l+\varepsilon)^{-p}, \forall\, p \geq m+2 A = u_{m+1}\,(l+\varepsilon)^{m+1}.$$

Since, $\Sigma\,(l+\varepsilon)^{-p}$ is a divergent series (being a geometric series with common ratio $(l+\varepsilon)^{-1}$, which is certainly greater then unity), then by comparison test, it follows that $\Sigma\, u_n$ diverges.

Case (iii) Let $l=1$.

Now, first consider the harmonic series $1+\dfrac{1}{2}+\dfrac{1}{3}+\ldots+\dfrac{1}{n}+\ldots$

Then $$\frac{u_n}{u_{n+1}}=\frac{n+1}{n}=1+\frac{1}{n} \Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=1.$$

Since, the harmonic series is divergent, we find that if $l=1$, a series may diverge.

Now, consider the series $\dfrac{1}{1^2}+\dfrac{1}{2^2}+\ldots+\dfrac{1}{n^2}+\ldots$

Then $$\frac{u_n}{u_{n+1}}=\frac{(n+1)^2}{n^2}=\left(1+\frac{1}{n}\right)^2 \Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=1.$$

Since, the series $\Sigma \frac{1}{n^2}$ converges, we find that if $l = 1$, a series may converge.

(b) Let us suppose $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = +\infty$ then there exist positive integers m and p such that

$$\frac{u_n}{u_{n+1}} > p \ \forall\, n \geq m, p > 1.$$

Replacing n by $m, m+1, m+2, \ldots, n-1$, we have

$$\frac{u_m}{u_{m+1}} > p$$

$$\frac{u_{m+1}}{u_{m+2}} > p$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\frac{u_{n-1}}{u_n} > p$$

Multiplying the corresponding sides of the above inequalities, we have

$$\frac{u_m}{u_n} > p^{n-m}$$

$$\Rightarrow \qquad u_n < p^{m-n} . u_m ,$$

$$\Rightarrow \qquad u_n < A . p^{-n} \ \forall\, n > m \text{ and } A = p^m u_m .$$

Since Σp^{-n} is convergent, then by comparison test the series Σu_n is convergent.

Q.8. Test the convergence of the series $\frac{(a+x)}{1!} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + \ldots$

Ans. Here, we have $u_n = \frac{(a+nx)^n}{n!}$

$$\Rightarrow \qquad u_{n+1} = \frac{[a+(n+1)x]^{n+1}}{(n+1)!} \qquad \Rightarrow \qquad \frac{u_n}{u_{n+1}} = \frac{\left[1+\frac{a/x}{n}\right]^n}{\left[1+\frac{1}{n}\right]^n \left[1+\frac{a/x}{n+1}\right]^{n+1}} \cdot \frac{1}{x}$$

$$\Rightarrow \qquad \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \left[\frac{\left[1+\frac{a/x}{n}\right]^n}{\left[1+\frac{1}{n}\right]^n \left[1+\frac{a/x}{n+1}\right]^{n+1}} \cdot \frac{1}{x} \right] = \frac{e^{a/x}}{x . e . e^{a/x}} = \frac{1}{ex}.$$

Hence, by D'alembert's ratio test the given series is convergent if $\frac{1}{ex} > 1$, *i.e.*, $x < \frac{1}{e}$ and divergent if $x > \frac{1}{e}$ and the test fails if $x = \frac{1}{e}$.

In this case

$$\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right)=\lim_{n\to\infty} n\log\left[\frac{\left[1+\frac{ae}{n}\right]^n}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{ae}{n+1}\right)^{n+1}}\right]$$

$$=\lim_{n\to\infty} n\left[n\log\left(1+\frac{ae}{n}\right)+\log e-n\log\left(1+\frac{1}{n}\right)-(n+1)\log\left(1+\frac{ae}{n+1}\right)\right]$$

$$=\lim_{n\to\infty} n\left[n\left(\frac{ae}{n}-\frac{a^2e^2}{2n^2}+\frac{a^3e^3}{3n^3}\dots\right)+1-\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\right)\right.$$

$$\left.-(n+1)\left(\frac{ae}{n+1}-\frac{a^2e^2}{2(n+1)^2}+\frac{a^3e^3}{3(n+1)^3}\right)\right]$$

$$=\lim_{n\to\infty}\left[-\frac{a^2e^2}{2}+\frac{1}{2}+\frac{a^2e^2}{2\left(1+\frac{1}{n}\right)}+\text{terms containing } n \text{ in the denominator}\right]$$

$$=-\frac{a^2e^2}{2}+\frac{1}{2}+\frac{a^2e^2}{2}=\frac{1}{2}<1.$$

Hence, by logarithmic test, the series is divergent.

Thus the given series is convergent if $x<\frac{1}{e}$ and divergent if $x\geq\frac{1}{e}$.

Q.9. Write and prove DeMorgan's and Bertrand's Test.

Ans. DeMorgan's and Bertrand's Test : The series $\Sigma\, u_n$ of positive terms is convergent or divergent according as

$$\lim\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]>1 \text{ or} <1.$$

Proof : Let $\lim\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]=k$, where $k>1$.

Take a number p such that $k>p>1$.

Compare the series $\Sigma\, u_n$ with the auxiliary series $\Sigma\, v_n$, where $v_n=\frac{1}{n(\log n)^p}$, which is convergent as $p>1$. The series $\Sigma\, u_n$ is convergent if after some particular term

$$\frac{u_n}{u_{n+1}}>\frac{v_n}{v_{n+1}}, \qquad \text{[By article 7, sixth from of the comparison test]}$$

i.e., $$\frac{u_n}{u_{n+1}}>\frac{1}{n(\log n)^p}.(n+1)\{\log(n+1)\}^p, \qquad \left[\because v_n=\frac{1}{n(\log n)^p}\right]$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(\frac{n+1}{n}\right)\left[\frac{\log\{n(1+1/n)\}}{\log n}\right]^p$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[\frac{\log n + \log(1+1/n)}{\log n}\right]^p$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[\frac{\log n + \frac{1}{n} - \frac{1}{2n^2} + \ldots}{\log n}\right]^p$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[1 + \frac{1}{n\log n} - \frac{1}{2n^2\log n} + \ldots\right]^p$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[1 + \frac{p}{n\log n} + \ldots\right]$$

i.e., $$\frac{u_n}{u_{n+1}} > 1 + \frac{1}{n} + \frac{p}{n\log n} + \ldots$$

i.e., $$n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1 + \frac{p}{\log n} + \ldots$$

i.e., $$n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 > \frac{p}{\log n} + \ldots$$

i.e., $$\left[n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right]\log n > p + \text{terms containing } n \text{ or } \log n$$

in the denominator.

Now as n becomes sufficiently large the L.H.S. and R.H.S. of (1) respectively approach k and p. Also $k > p$.

Thus (1) is satisfied for sufficiently large values of n.

Hence the series $\Sigma\, u_n$ is convergent if

$$\lim\left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n\right] > 1.$$

Similarly, it can be proved as in the case of Raabe's test that $\Sigma\, u_n$ is divergent if

$$\lim\left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n\right] < 1.$$

❑

Part-B : Integral Calculus

UNIT V

SECTION-A (VERY SHORT ANSWER TYPE) QUESTIONS

Q.1. Define partition of a losed interval.

Ans. Let $I=[a,b]$ be a closed and bounded interval. Then a finite set of points $P=\{x_0, x_1, x_2, \ldots, x_n\}$ such that $a=x_0<x_1<x_2 \ldots <x_{n-1}<x_n=b$ is called a partition or division of the interval $I=[a,b]$.

Q.2. Define upper and lower intergrals.

Ans. The infimum of the set of the upper sums is called the upper integral of f over $[a,b]$ and is denoted by $U=\int_a^{-b} f(x)\,dx$. Also, the supremum of the set of the lower sums is called the lower integral of over $[a,b]$ and is denoted by $L=\int_{-a}^{b} f(x)\,dx$.

Q.3. Define Riemann integral.

Ans. A bounded function f is said to be Riemann integrable, or simply integrable over $[a,b]$, if its upper and lower integrals are equal; and their common value being called Riemann integral or simply the integral denoted by $\int_a^b f(x)\,dx$.

Q.4. Define lower and upper Riemann integrals.

Ans. If f is bounded on the interval $[a,b]$, then for every $P \in \mathbf{P}(a,b)$, $U(P,f)$ and $L(P,f)$ exist and are bounded. Then the lower Riemann integral is defined as $\int_{-a}^{b} f=\sup_{\mathbf{P}} L(P,f)$ and the upper Riemann integral is defined as $\int_a^{-b} f=\inf_{\mathbf{P}} U(P,f)$.

Q.5. Find $L(P,f)$ and $U(P,f)$ if $f(x)=x$, for $x\in[0,3]$ and let $P=[0,1,2,3]$ be the partition of [0, 3].

Ans. Let partition P divided the interval [0, 3] into the subinterval $I_1=[0,1]$, $I_2=[1,2]$ and $I_3=[2,3]$.

Then length of these intervals are given by

$$\delta_1=1-0=1$$
$$\delta_2=2-1=1$$
$$\delta_3=3-2=1$$

Let M_r and m_r be respectively the *l.u.b.* and *g.l.b.* of the function f in $[x_{r-1}, x_r]$, then we get

$$M_1=1, m_1=0, M_2=2, m_2=1, M_3=3 \text{ and } m_3=2$$

Therefore $U(P,f)=\sum_{r=1}^{3} M_r\,\delta_r = M_1\,\delta_1 + M_2\,\delta_2 + M_3\,\delta_3$

$= 1.1 + 2.1 + 3.1 = 1 + 2 + 3 = 6$

and $L(P,f)=\sum_{r=1}^{3} m_r\,\delta_r = m_1\,\delta_1 + m_2\,\delta_2 + m_3\,\delta_3$

$= 0.1 + 1.1 + 2.1 = 0 + 1 + 2 = 3.$

Q.6. Compute $\int_1^2 x^3\,dx$.

Ans. Let $f(x)=x^3, 1 \le x \le 2$. Then f is continuous on [1, 2]. Moreover, if $\phi(x)=x^4/4\ (1 \le x \le 2)$, then $\phi'(x)=x^3=f(x), (1 \le x \le 2)$.

Hence by the fundamental theorem of integral calculus, we have

$$\int_1^2 x^3\,dx = \phi(2)-\phi(1) = \frac{2^4}{4}-\frac{1^4}{4}=\frac{15}{4}.$$

Q.7. Find the upper and lower Riemann integrals for the function f defined on [0, 1] as follows :

$$f(x)=\begin{cases} \sqrt{(1-x^2)}, & \text{when } x \text{ is rational} \\ (1-x), & \text{when } x \text{ is irrational} \end{cases}$$

Ans. We have $(1-x^2)-(1-x)^2 = 2x(1-x) > 0\ \forall\ x \in [0,1]$

Therefore, $m_r = (1-x)$ and $M_r = \sqrt{1-x^2}$.

Now, $$\int_{-0}^{1} f = \int_0^1 (1-x)\,dx = \left[x-\frac{x^2}{2}\right]_0^1 = 1-\frac{1}{2}=\frac{1}{2}$$

and $$\int_0^{-1} f = \int_0^1 \sqrt{(1-x^2)}\,dx = \left[\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_0^1 = \frac{1}{2}\sin^{-1}1 = \frac{1}{2}.\frac{\pi}{2}=\frac{\pi}{4}.$$

Clearly, $\int_0^{-1} f \ne \int_{-0}^{1} f$

Hence, $f \notin \mathbf{R}[0,1]$.

Q.8. Show that the Bonnet's mean value theorem does not hold on $[-1,1]$ for $f(x)=g(x)=x^2$.

Ans. The function $f(x)=x^2$ is not monotonic on $[-1,1]$ since for the interval $[-1,0]$ it is non-increasing and for [0, 1] it is non-decreasing. Thus the conditions of the Bonnet's mean value theorem are not satisfied and hence the theorem does not hold in $[-1,1]$.

Q.9. Find $\int_1^2 x^3\,dx$, using fundamental theorem of integral calculus.

Ans. Here, we have $f(x)=x^3,\ 1 \le x \le 2$

Clearly f is continuous on $[1, 2]$

Now, if $\phi(x)=\frac{x^4}{4}\ (1 \le x \le 2)$

Then $\phi(x)=x^3=f(x)$

Therefore, by fundamental theorem of integral calculus; we have

$$\int_1^2 x^3\, dx=\phi(2)-\phi(1)=\frac{2^4}{4}-\frac{1^4}{4}=\frac{15}{4}.$$

Q.10. Show that $\displaystyle\int_0^\infty \frac{dx}{(x^2+a)^{n+1}}=\frac{\pi}{2}.\frac{1.3 \le (2n-1)}{2^n\, n!a^{(2n+1)/2}}.$

Ans. We have $\displaystyle\int_0^\infty \frac{dx}{x^2+a}=\frac{1}{\sqrt{a}}\left[\tan^{-1}\frac{x}{\sqrt{a}}\right]_0^\infty=\frac{1}{\sqrt{a}}.\frac{\pi}{2}=\frac{\pi}{2}a^{-1/2}$

Differentiating both sides n times w.r.t. 'a', we get

$$\int_0^\infty \frac{(-1)^n\, n!}{(x^2+a)^{n+1}}dx=\frac{\pi}{2}\frac{(-1)^n\, 1.3.5.\le(2n-1)}{2^n\, a^{(2n+1)/2}}$$

or $$\int_0^\infty \frac{dx}{(x^2+a)^{n+1}}=\frac{\pi}{2}\frac{1.3.5.\le(2n-1)}{2^n\, n!\, a^{(2n+1)/2}}$$

Q.11. If the function $f(x)$ defined by

$$f(x)=\begin{cases}0, & \text{when } x \text{ is and integer}\\ 1, & \text{when } x \text{ is not an integer}\end{cases}$$

Show that $f(x)$ is R-integrable in every interval.

Ans. Consider an arbitrary interval $[0, a]$, where $a>0, a \in Z$. Then clearly, the function $f(x)$ is continuous at all points in the interval excepts at points $x=1, 2, 3, \ldots, a$, because it is given that $f(x)=0$ when x is an integer and

$$f(x-0)=1=f(x+0)$$

Therefore, the given function $f(x)$ has a finite number of discontinuities at $x=1, 2, 3, \ldots a$ in the interval $[0, a]$. Thus, if each of these points of discontinuity be enclosed in an interval whose length is less than ε/a, then all these points will be enclosed in a non ... overlapping intervals whose total length is less than $\frac{\varepsilon}{a}.a$ *i.e.,* less than ε.

Hence, the given function $f(x)$ is integrable in the interval $[0, a]$.

Q.12. Evaluate the following $\displaystyle\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\right].$

Ans. The general term is given by $(r^{th}\text{ term})=\frac{1}{n+r}.$

We have to find $\displaystyle\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}=\lim_{n\to\infty}\frac{1}{n[1+r/n]}=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+(r/n)}.$

Since the limit of r in the summation are 1 to n, therefore the lower limit of integration $= \lim_{n \to \infty} \frac{1}{n} = 0.$

Also, the upper limit of integration $= \lim_{n \to \infty} \frac{n}{n} = 1.$

Hence, the required limit $\int_0^1 \frac{1}{a+x} dx = [\log(1+x)]_0^1 = \log 2$

Q.13. Define Riemann integrable function.

Ans. A real valued function $f(x)$ is said to be Riemann integrable on $[a, b]$ if and only if their lower and upper Riemann integrals are equal.

i.e., $\quad$ *iff* $\int_{-a}^{b} f = \int_{a}^{-b} f$

The common value of these integrals is known as the Riemann integral of f on $[a, b]$ and is denoted by $\int_a^b f(x)\,dx$.

i.e., $\quad \int_a^b f(x) = \int_{-a}^{b} f(x) = \int_a^{-b} f(x)\,dx$

SECTION-B SHORT ANSWER TYPE QUESTIONS

Q.1. Let $f(x) = x, 0 \le x \le 1$ and lets $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ be a partition of [0,1], find $U(P, f)$ and $L(P, f)$.

Ans. Let the partition P divides the interval [0, 1] into the subintervals

$$I_1 = \left[0, \frac{1}{4}\right], I_2 = \left[\frac{1}{4}, \frac{1}{2}\right], I_3 = \left[\frac{1}{2}, \frac{3}{4}\right], I_4 = \left[\frac{3}{4}, 1\right]$$

Clearly, the length of each subinterval is $\frac{1}{4}$.

Now, let M_r and m_r respectively be the *l.u.b.* and *g.l.b.* of the function f in $[x_{r-1}, x_r]$, then we get

$$M_1 = \frac{1}{4}, M_2 = \frac{1}{2}, M_3 = \frac{3}{4}, M_4 = 1 \quad \text{and} \quad m_1 = 0, m_2 = \frac{1}{4}, m_3 = \frac{1}{2}, m_4 = \frac{3}{4}.$$

Therefore,
$$U(P, f) = \sum_{r=1}^{4} M_r\,\delta_r = M_1\,\delta_1 + M_2\,\delta_2 + M_3\,\delta_3 + M_4\,\delta_4$$
$$= \frac{1}{4}.\frac{1}{4} + \frac{1}{2}.\frac{1}{4} + \frac{3}{4}.\frac{1}{4} = 1.\frac{1}{4} = \frac{1}{16} + \frac{1}{8} + \frac{3}{16} + \frac{1}{4} = \frac{5}{8}.$$

and
$$L(P, f) = \sum_{r=1}^{4} m_r\,\delta_r = m_1\,\delta_1 + m_2\,\delta_2 + m_3\,\delta_3 + m_4\,\delta_4$$
$$= 0.\frac{1}{4} + \frac{1}{4}.\frac{1}{4} + \frac{1}{2}.\frac{1}{4} + \frac{3}{4}.\frac{1}{4}$$
$$= 0 + \frac{1}{16} + \frac{1}{8} + \frac{3}{16} = \frac{3}{8}.$$

Q.2. Let $f(x)=x$ on [0, 1]. Find $\int_{-0}^{1} x\,dx$, and $\int_{-0}^{1} x\,dx$, by partitioning [0, 1] into n equal parts. Also, show that $f \in R[0,1]$.

Ans. Let the partition P divides the interval $[0,1]$ into n subinterval such that

$$P=\left\{0,\frac{1}{n},\frac{2}{n},\ldots,\frac{r-1}{n},\frac{r}{n},\ldots,\frac{n}{n}=1\right.$$

Clearly, here we have $m_r=\dfrac{r-1}{n}, M_r=\dfrac{r}{n}$ and $\delta_r=\dfrac{1}{n}$ for $r=1,2,\ldots,n$

Now, by definition, we have

$$L[P,f]=\sum_{r=1}^{n} m_r\delta_r=\sum_{r=1}^{n}\frac{r-1}{n}.\frac{1}{n}=\frac{1}{n^2}\sum_{r=1}^{n}(r-1)$$

$$=\frac{1}{n^2}[1+2+3+\ldots+(n-1)]=\frac{(n-1).n}{2n^2}=\frac{n-1}{2n}$$

and

$$U[P,f]=\sum_{r=1}^{n} M_r\delta_r=\sum_{r=1}^{n}\frac{r}{n}.\frac{1}{n}$$

$$=\frac{1}{n^2}\sum_{r=1}^{n} r=\frac{1}{n^2}[1+2+3+\ldots+n]=\frac{n(n+1)}{2n^2}=\frac{n+1}{2n}$$

Therefore, $$\int_{-0}^{1} x\,dx=\lim_{\|P\|\to 0} L(P,f)=\lim_{n\to\infty}\frac{n-1}{2n}=\frac{1}{2}$$

and $$\int_{0}^{-1} x\,dx=\lim_{\|P\|\to 0} U(P,f)=\lim_{n\to\infty}\frac{n+1}{2n}=\frac{1}{2}$$

From above, it is clear that

$$\int_{-0}^{1} x\,dx=\int_{0}^{-1} x\,dx$$

Hence, $$\int_{0}^{1} x\,dx=\frac{1}{2}.$$

Q.3. Show that if f is defined on $[a,b]$ by $f(x)=k\ \forall\ x\in[a,b]$ where k is a constant, then $f\in R[a,b]$ and $\int_{a}^{b} k=k(b-a)$.

Ans. Obviously the given function is bounded over $[a,b]$.

Let $P=\{a=x_0,x_1,x_2,\ldots,x_n=b\}$ be any partition of $[a,b]$. Then for any subinterval $[x_{r-1},x_r]$, we have $m_r=k, M_r=k$.

Now, $$U(P,f)=\sum_{r=1}^{n} M_r\,\Delta x_r=\sum_{r=1}^{n} k\,\Delta x_r=k\sum_{r=1}^{n}\Delta x_r=k[\Delta x_1+\Delta x_2+\ldots+\Delta x_n]$$

$$=k[(x_1-x_0)+(x_2-x_1)+\ldots+(x_n-x_{n-1})]=k(x_n-x_0)=k(b-a)$$

and $$L(P,f)=\sum_{r=1}^{n} m_r\,\Delta x_r=\sum_{r=1}^{n} k\,\Delta x_r=k(b-a).$$

Hence $\int_a^{-b} f = \inf U(P, f) = \inf \{k(b-a)\} = k(b-a)$

and $\int_{-a}^{b} f = \sup L(P, f) = \sup \{k(b-a)\} = k(b-a)$.

Thus $\int_a^{-b} f = \int_{-a}^{b} f = k(b-a)$. Hence $f \in R[a, b]$ and $\int_a^b f = k(b-a)$.

Q.4. If a function f is defined on $[0, a], a > 0$ by $f(x) = x^3$, then show that f is Riemann integrable on $[0, a]$ and $\int_0^a f(x)\,dx = \frac{a^4}{4}$.

Ans. Let $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \ldots, \frac{(n-1)a}{n}, \frac{na}{n} = a\right\}$ be the partition of $[0, a]$ obtained by dessecting [0, a] into n equal parts. Then

$$I_r = r \text{ th sub-interval} = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$$

and $\Delta x_r = \text{length of } I_r = \frac{a}{n}, r = 1, 2, \ldots, n.$

Let m_r and M_r be respectively the infimum and supremum of f in I_r.

Since $f(x) = x^3$ is an increasing function in $[0, a]$, therefore

$$m_r = \frac{(r-1)^3 a^3}{n^3} \text{ and } M_r = \frac{r^3 a^3}{n^3}, r = 1, 2, \ldots, n.$$

Now $$L(P, f) = \sum_{r=1}^{n} m_r \,\Delta x_r = \sum_{r=1}^{n} \left[\frac{(r-1)^3 a^3}{n^3} \cdot \frac{a}{n}\right] = \frac{a^4}{n^4} \sum_{r=1}^{n} (r-1)^3$$

$$= \frac{a^4}{n^4}[1^3 + 2^3 + \ldots + (n-1)^3] = \frac{a^4}{n^4} \cdot \left[\frac{(n-1)n}{2}\right]^2 = \frac{a^4}{4}\left(1 - \frac{1}{n}\right)^2.$$

$$\therefore \int_{-0}^{a} f(x)\,dx = \lim_{n \to \infty} L(P, f) = \lim_{n \to \infty} \frac{a^4}{4}\left(1 - \frac{1}{n}\right)^2 = \frac{a^4}{4}.$$

Again $$U(P, f) = \sum_{r=1}^{n} M_r \,\Delta x_r = \sum_{r=1}^{n} \left[\frac{r^3 a^3}{n^3} \cdot \frac{a}{n}\right] = \frac{a^4}{n^4} \sum_{r=1}^{n} r^3$$

$$= \frac{a^4}{n^4}(1^3 + 2^3 + \ldots + n^3) = \frac{a^4}{n^4} \cdot \left[\frac{n(n+1)}{2}\right]^2 = \frac{a^4}{4}\left(1 + \frac{1}{n}\right)^2.$$

$$\therefore \int_0^{-a} f(x)\,dx = \lim_{n \to \infty} U(P, f) = \lim_{n \to \infty} \frac{a^4}{4}\left(1 + \frac{1}{n}\right)^2 = \frac{a^4}{4}.$$

Since $\int_{-0}^{a} f = \int_0^{-a} f$, f is Riemann integrable on $[0, a]$ and $\int_0^a f(x)\,dx = \int_0^a x^3\,dx = \frac{a^4}{4}$.

Q.5. Show that the function $f(x) = \sin x$ is integrable on $\left[0, \frac{\pi}{2}\right]$.

Ans. Consider the partition

$$P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \ldots, \frac{(r-1)\pi}{2n} \ldots, \frac{r\pi}{2n}, \ldots, \frac{n\pi}{2n} = \frac{\pi}{2}\right\}$$

which is obtained by dividing $\left(0, \frac{\pi}{2}\right)$ into n equal parts with length of each subinterval $= \frac{\pi}{2n}$.

Let $I_r = \left[\frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}\right]$ be the r^{th} subinterval.

Now, since $f(x) = \sin x$ is increasing in $\left[0, \frac{\pi}{2}\right]$, therefore

$$m_r = \sin \frac{(r-1)\pi}{2n} \text{ and } M_r = \sin \frac{r\pi}{2n}, \ r = 1, 2, \ldots, n$$

Now $\quad U(P, f) - L(P, f) = \sum_{r=1}^{n} (M_r - m_r)\delta_r$

$$= \sum_{r=1}^{n} \left[\sin \frac{r\pi}{2n} - \sin \frac{(r-1)\pi}{2n}\right] \frac{\pi}{2n} = \left[\sin \frac{n\pi}{2n} - 0\right] \frac{\pi}{2n} = \frac{\pi}{2n}$$

For given $\varepsilon > 0$ there exist $m \in M$ such that $\frac{\pi}{2n} < \varepsilon \ \forall \ n \geq m$. Therefore, for a given there exists a partition P of $\left[0, \frac{\pi}{2}\right]$ such that $U(P, f) - L(P, f) < \varepsilon$

Hence, the function $f(x) = \sin x$ is R-integrable.

Q.6. If $f(x) = x + x^2$ for rational values of x in the interval $[0, 2]$ and $f(x) = x^2 + x^3$ for irrational values of x in the same interval, evaluate the upper and the lower Riemann integrals of f over $[0, 2]$.

Ans. We have $(x + x^2) - (x^2 + x^3) = x - x^3 = x(1 - x^2)$

so that $\quad (x + x^2) - (x^2 + x^3) > 0$ if $0 < x < 1$ and < 0 if $1 < x < 2$.

If P is any partition of $[0, 2]$, then any subinterval of P, however small it may be, will contain rational as well as irrational points.

With usual notations, we have for all values of r

$$M_r = x + x^2, \text{ if } 0 < x < 1; M_r = x^2 + x^3, \text{ if } 1 < x < 2$$

and $$m_r = x^2 + x^3, \text{ if } 0 < x < 1; m_r = x + x^2, \text{ if } 1 < x < 2.$$

Hence, $\int_0^{-2} f(x)\,dx = \int_0^1 (x + x^2)\,dx + \int_1^2 (x^2 + x^3)\,dx = \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_1^2$

$$= \left(\frac{1}{2} + \frac{1}{3}\right) - 0 + \left(\frac{8}{3} + \frac{16}{4}\right) - \left(\frac{1}{3} + \frac{1}{4}\right) = \frac{83}{12} = 6\frac{11}{12}$$

and $$\int_{-0}^{2} f(x)\,dx = \int_0^1 (x^2+x^3)\,dx + \int_1^2 (x+x^2)\,dx$$

$$=\left[\frac{x^3}{3}+\frac{x^4}{4}\right]_0^1+\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_1^2=\frac{1}{3}+\frac{1}{4}+\frac{4}{2}+\frac{8}{3}-\frac{1}{2}-\frac{1}{3}=\frac{53}{12}=4\frac{5}{12}.$$

Q.7. Let function f be defined on $[0,1]$ by

$$f(x)=\frac{1}{n} \text{ for } \frac{1}{n+1}<x\le\frac{1}{n}, n\in N$$
$$=0 \text{ for } x=0.$$

Show that $f\in R[0,1]$. Also find $\int_0^1 f(x)\,dx$.

Ans. Clearly, the points of discontinuity of f are $\frac{1}{2},\frac{1}{3},\frac{1}{4}$.

Also the set of points of discontinuity of f has only one limit point at $x=0$. Then $f\in R[0,1]$. [using theorem 4]

Now $$\int_0^1 f(x)\,dx = \lim_{n\to\infty}\sum_{r=1}^{n}\int_{1/(r+1)}^{1/r}\frac{1}{r}\,dx=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r}\left[\frac{1}{r}-\frac{1}{r+1}\right]$$

$$=\lim_{n\to\infty}\left[\left(1-\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\frac{1}{n}\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$=\lim_{n\to\infty}\left[\left(1-\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}\right)-\left(\frac{1}{2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}\right)\right]$$

$$=\lim_{n\to\infty}\left[\left(1+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right)-\left(\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}\ldots+\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$=\lim_{n\to\infty}\left[\left(1+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right)-\left(1-\frac{1}{n+1}\right)\right]$$

The series $1+\frac{1}{2^2}+\ldots+\frac{1}{n^2}+$ converges to $\frac{\pi^2}{6}$, therefore

$$\int_0^1 f(x)\,dx=\lim_{n\to\infty}\left(1+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right)-1+\lim_{n\to\infty}\frac{1}{n+1}=\frac{\pi^2}{6}-1$$

Q.8. State and prove Darboux theorem.

Ans. Darboux Theorem : Let f be a bounded function defined on $[a,b]$. Then to every $\varepsilon>0$, there corresponds $\delta>0$ such that

$$U(P,f)<\int_a^{-b} f+\varepsilon \quad\text{and}\quad L(P,f)>\int_{-a}^{b} f-\varepsilon$$

for all partitions P with $\|P\|\le\delta$.

Proof : Let $\varepsilon>0$ be given. Since $\int_a^{-b} f$ is the infimum of $U(P,f)$ and $\int_{-a}^{b} f$ is the supremum of $L(P,f)$ for all partitions P, therefore, for given $\varepsilon>0$ there exist partitions P_1 and P_2 such that

$$U(P_1, f) < \int_a^{-b} f + \varepsilon \qquad ...(1)$$

and $$L(P_2, f) > \int_{-a}^{b} f - \varepsilon. \qquad ...(2)$$

Let P_3 be the common refinement of P_1 and P_2. Then by theorem 3 of article 2, we get

$$U(P_3, f) \le U(P_1, f) \text{ and } L(P_3, f) \ge L(P_2, f). \qquad ...(3)$$

Therefore, from (1), (2) and (3), we get

$$U(P, f) < \int_a^{-b} f + \varepsilon \text{ and } L(P, f) > \int_{-a}^{b} f - \varepsilon$$

for all partitions P of $[a, b]$ with $\|P\| \le \delta$, where $\delta = \|P_3\| > 0$.

Q.9. Let f be a function on $[0,1]$ by

$$f(x) = \begin{cases} \frac{1}{2^n} & \text{for } \frac{1}{2^n + 1} < x \le \frac{1}{2^n}, n = 0, 1, 2 \\ 0 & \text{for } x = 0 \end{cases}$$

Show that $f \in R[0,1]$. Also find the value of $\int_0^x f(t)\,dt$.

Ans. Here, we may defined the function $f(x)$ as follows :

$$f(x) = 1 \text{ if } \frac{1}{2} < x \le 1 = \frac{1}{2} \text{ if } \frac{1}{2^2} < x \le \frac{1}{2} = \frac{1}{2^2} \text{ if } \frac{1}{2^3} < x \le \frac{1}{2^2} = \frac{1}{2^{n-1}} \text{ if } \frac{1}{2^n} < x \le \frac{1}{2^{n-1}}$$

and $f(0) = 0.$

Clearly $|f(x)| \le 1, \forall x \in [0,1]$, therefore, $f(x)$ is bounded on [0, 1]. But f is not continuous on [0, 1]. The set of points of discontinuity of f in [0, 1] is $\left\{0, \frac{1}{2}, \frac{1}{2^2}, ..., \frac{1}{2^n}\right\}$ which is an infinite set and 0 is the only limit point of this set. Therefore $f \in R[0,1]$.

Also, we have

$$\int_0^x f(x)\,dx \approx \int_{1/2^m}^{x} f + \int_{1/2^{m+1}}^{1/2^m} f + \int_{1/2^{m+2}}^{1/2^{m+1}} f +$$

$$= \int_{1/2^n}^{x} \frac{1}{2^{m-1}} \int_{1/2^{m+1}}^{1/2^m} \frac{1}{2^m} + \int_{1/2^{m+2}}^{1/2^{m+1}} \frac{1}{2^{m+1}} +$$

$$= \frac{1}{2^{m-1}}\left[x - \frac{1}{2^m}\right] + \frac{1}{2^m}\left[\frac{1}{2^m} - \frac{1}{2^{m+1}}\right] + \frac{1}{2^{m+1}}\left[\frac{1}{2^{m+1}} - \frac{1}{2^{m+2}}\right] +$$

$$= \frac{1}{2^{m-1}}\left[x - \frac{1}{2^m}\right] + \frac{1}{2^{2m+1}} + \frac{1}{2^{2m+3}} + \frac{1}{2^{2m+5}} +$$

$$= \frac{x}{2^{m-1}} - \frac{1}{2^{2m+1}} + \frac{1/2^{m+1}}{1 - 1/4} \qquad \left[\text{Sum of infinite G.P.} = \frac{a}{1-r}\right]$$

$$= \frac{x}{2^{m-1}} - \frac{1}{3.2^{2m-2}}$$

Q.10. Let f be the function defined on [0, 1] by

$$f(x)=\begin{cases}0 & \text{when } x \text{ is irrational,}\\ 1 & \text{when } x \text{ is rational.}\end{cases}$$

Calculate $\int_{-0}^{1} f$ and $\int_{0}^{-1} f$ and hence show that $f \notin R[0,1]$.

Ans. First, we observe that f is bounded, for evidently

$$0 \le f(x) \le 1 \; \forall \; x \in [0,1].$$

Let P be any partition of [0, 1]. Then for any subinterval $[x_{r-1}, x_r]$ of P, we have $m_r = 0$ and $M_r = 1$, because every subinterval will contain rational as well as irrational numbers. Note that rational as well as irrational points are everywhere dense.

It follows that $L(P,f) = \sum_{r=1}^{n} m_r \, \Delta x_r = \sum_{r=1}^{n} 0 . \Delta x_r = 0$

and $$U(P,f) = \sum_{r=1}^{n} M_r \, \Delta x_r = \sum_{r=1}^{n} 1 . \Delta x_r = \sum_{r=1}^{n} \Delta x_r = 1.$$

$\therefore$ $$\int_{-0}^{1} f = \lim_{n\to\infty} L(P,f) = 0 \text{ and } \int_{0}^{-1} f = \lim_{n\to\infty} U(P,f) = 1.$$

Since $\int_{-0}^{1} f \neq \int_{0}^{-a} f$, we have $f \notin R[0,1]$.

Q.11. Write and prove fundamental theorem of integral calculus.

Ans. Theorem (Fundamental theorem of Integral Calculus) : Let $f \in R[a,b]$ and let ϕ be a differentiable function on $[a,b]$ such that $\phi'(x) = f(x)$ for all $x[a,b]$. Then $\int_{a}^{b} f(x)\,dx = \phi(b) - \phi(a)$.

Proof : Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a,b]$. Now ϕ is differentiable on $[a,b]$ implies that ϕ is differentiable on each subinterval $[x_{r-1}, x_r]$.

Hence by the mean value theorem of differential calculus, we find that there exists ξ_r in $[x_{r-1}, x_r]$, $r = 1, 2, \ldots, n$, such that

$$\phi(x_r) - \phi(x_{r-1}) = (x_r - x_{r-1})\phi'(\xi_r) = (\phi)'(\xi_r) . \Delta x_r$$

or $$\phi(x_r) - \phi(x_{r-1}) = f(\xi_r) . \Delta x_r \qquad [\because \phi'(\xi_r) = f(\xi_r)]$$

or $$\sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})] = \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r . \qquad \ldots(1)$$

Now $$\sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})] = \phi(x_1) - \phi(x_0) + \phi(x_2) - \phi(x_1) + \ldots$$

$$\ldots + \phi(x_n) - \phi(x_{n-1})$$

$$= \phi(x_n) - \phi(x_0) = \phi(b) - \phi(a).$$

Ir follows from (1) that

$$\phi(b) - \phi(a) = \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r \qquad \ldots(2)$$

Taking limit as $||P|| \to 0$,

we get $$\phi(b)-\phi(a)=\int_a^b f(x)\,dx,$$

since $\sum_{r=1}^{n} f(\xi_r)\Delta x_r$ tends to $\int_a^b f(x)\,dx$ as $\|P\|\to 0$.

The result of the above theorem is usually written in the form

$$\int_a^b \phi'(x)\,dx=\phi(b)-\phi(a).$$

Q.12. State and prove second mean value theorem.

Ans. Theorem (Second Mean Value Theorem) : Let

$f\in \mathbf{R}[a,b]$ and $g\in \mathbf{R}[a,b]$ and $g(x)\geq 0$ or $\leq 0\ \forall\, x\in[a,b]$.

Then there exists a number μ with $m\leq\mu\leq M$ such that

$$\int_a^b f(x)g(x)\,dx=\mu\int_a^b g(x)\,dx$$

where *m, M* are the bounds of *f* on $[a,b]$.

Proof : First let $g(x)\geq 0\ \forall\, x\in[a,b]$.

Then $$mg(x)\leq f(x)g(x)\leq Mg(x)\ \forall\, x\in[a,b].$$

If follows from Cor. 5 of theorem 1 of article 8 that

$$m\int_a^b g(x)\,dx\leq\int_a^b f(x)g(x)\,dx\leq M\int_a^b g(x)\,dx$$

or $$m\int_a^b g(x)\,dx\geq\int_a^b f(x)g(x)\,dx\geq M\int_a^b g(x)\,dx$$

according as $a\leq b$ or $a\geq b$.

Hence there exists a number μ with $m\leq\mu\leq M$ such that

$$\int_a^b f(x)g(x)\,dx=\mu\int_a^b g(x)\,dx.$$

Now let $g(x)\leq 0\ \forall\, x\in[a,b]$.

Then $-g(x)\geq 0\ \forall\, x\in[a,b]$.

Hence by the above result for some $\mu\in[m,M]$, we have

$$\int_a^b f(x)[-g(x)]\,dx=\mu\int_a^b[-g(x)]\,dx \quad\text{or}\quad \int_a^b f(x)g(x)\,dx=\mu\int_a^b g(x)\,dx.$$

Q.13. From the definition of a definite integral as the limit of a sum, evaluate $\int_a^b e^x\,dx$.

Ans. Here, we have $f(x)=e^x$.

Therefore $$f(a)=e^a$$
$$f(a+h)=e^{a+h}$$
... ... etc.

Now $$\int_a^b e^x dx=\lim_{h\to 0} h[e^a+e^{a+h}+e^{a+2h}+\ldots+e^{a+(n-1)h}]$$

where, $nh = b - a$ and $n \to \infty$ as $h \to 0$

$$= \lim_{h\to 0} he^a[1+e^h+e^{2h}+\ldots+e^{(n-1).h}]$$

$$= \lim_{h\to 0} he^a\left\{\frac{(e^h)^n-1}{e^h-1}\right\} = \lim_{h\to 0} he^a\left\{\frac{e^{nh}-1}{e^h-1}\right\}$$

$$= \lim_{h\to 0} he^a\left[\frac{e^{b-a}-1}{e^h-1}\right] \qquad [\because nh = b-a]$$

$$= \lim_{h\to 0} e^a\left[\frac{e^{b-a}-1}{\frac{e^h-1}{h}}\right] = e^b - e^a \qquad \left(\because \lim_{h\to 0}\frac{e^h-1}{h} = 1\right)$$

Q.14. Evaluate $\int_a^b x^2 dx$, directly from the definition of integral as the limit of a sum.

Ans. We know that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h[f(a)+f(a+h)+f(a+2h)+\ldots+f(a+(n-1)h\}] \qquad \ldots(1)$$

Here $f(x) = x^2$ $\qquad f(a) = a^2$

$f(a+h) = (a+h)^2$

.... and so on.

Put all these values in (1), we get

$$\int_a^b x^2 dx = \lim_{n\to\infty} h[a^2+(a+h)^2+(a+2h)^2+\ldots+\{a+(n-1)h\}^2]$$

where $h \to 0$ as $n \to \infty$ and $nh \to b - a$

$$= \lim_{h\to\infty} h[na^2+2ah\{1+2+3+\ldots+(n-1)\}]+h^2[1^2+2^2+\ldots+(n-1)^2].$$

Using $\quad \Sigma n = \dfrac{n(n+1)}{2}$ and $\Sigma n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$$\therefore \quad \int_a^b x^2\,dx = \lim_{h\to\infty} h\left[na^2+2ah\frac{(n-1)n}{2}+\frac{h^2}{6}(n-1)n(2n-1)\right]$$

$$= \lim_{h\to\infty}\left[(nh)a^2+a(nh)(n-1)h+\frac{1}{6}(nh)(n-1)h(2n-1)h\right]$$

$$= \lim_{h\to\infty}\left[(nh)a^2+a(nh)^2\left(1-\frac{1}{n}\right)+\frac{1}{6}2(nh^3)\left(1-\frac{1}{n}\right)\left(1-\frac{1}{2n}\right)\right]$$

$$= (b-a)a^2+a(b-a)^2+\frac{1}{3}(b-a)^3 \qquad (\because \text{as } n\to\infty, h\to 0, nh\to b-a)$$

$$=\frac{1}{3}(b-a)[3a^2+3(b-a)a+b^2-2ab+a^2]$$

$$=\frac{1}{3}(b-a)(a^2+ab+b^2)=\frac{1}{3}(b^3-a^3)$$

Q.15. (i) Taking $f(x)=x$, $g(x)=e^x$, verify the second mean value theorem in [–1, 1]. [Therefore 2 of article 10].

(ii) Also verify Bonnet's mean value theorem in [–1, 1] for the functions $f(x)=e^x$ and $g(x)=x$.

Ans. (i) Since f and g are continuous on $[-1,1]$, we have $f, g \in \boldsymbol{R}[-1,1]$.

Also $g(x)>0$ for all $x \in [-1,1]$. Hence the conditions of theorem 2 of article 10 are satisfied. Now

$$\int_{-1}^{1} f(x)g(x)\,dx = \int_{-1}^{1} xe^x\,dx = [xe^x - e^x]_{-1}^{1} = \frac{2}{e}. \quad ...(1)$$

and $$\int_{-1}^{1} g(x)\,dx = \int_{-1}^{1} e^x\,dx = [e^x]_{-1}^{1} = e - e^{-1} + \frac{e^2-1}{e}.$$

Since f is continuous on $[-1,1]$, it takes every value between $f(-1)=-1$ and $f(1)=1$. Let $\mu = 2/(e^2-1)$. Since $e>2$, we have $e^2>4 \Rightarrow e^2-1>3$ so that $0<\mu<1$.

It follows that there is a point ξ in $[-1,1]$ such that $f(\xi)=2/(e^2-1)$.

Accordingly, we have $f(\xi)\int_{-1}^{1} g(x)\,dx = \frac{2}{e^2-1}.\frac{e^2-1}{e} = \frac{2}{e}$. ...(2)

From (1) and (2), we have $\int_{-1}^{1} f(x)g(x)\,dx = f(\xi)\int_{-1}^{1} g(x)\,dx$.

Thus the second mean value theorem is verified.

(ii) Since $g(x)=x$ is continuous on [–1, 1] we have $g \in \boldsymbol{R}[-1,1]$.

Also $f(x)=e^x$ is monotonically non-decreasing and positive on $[-1,1]$. Hence all the conditions of the Bonnet's mean value theorem are satisfied. As in (i), we have

$$\int_{-1}^{1} f(x)g(x)\,dx = \int_{-1}^{1} e^x\, x\,dx = \frac{2}{e}$$

Now $$\int_{\eta}^{1} g(x)\,dx = \int_{\eta}^{1} x\,dx = \frac{1}{2}[1-\eta^2] \quad \therefore\ f(1)\int_{\eta}^{1} g(x)\,dx = \frac{e}{2}(1-\eta^2).$$

We choose η such that $\frac{2}{e} = \frac{e}{2}(1-\eta^2)$ *i.e.,* $\eta^2 = \frac{e^2-4}{e^2}$.

Also it is easy to see that $0<\eta<1$, where $\eta = \frac{\sqrt{(e^2-4)}}{e}$.

For this value of η, we then have $\int_{-1}^{1} f(x)g(x)\,dx = f(1)\int_{\eta}^{1} g(x)\,dx$.

Hence Bonnet's mean value theorem is verified.

Q.16. Evaluate $\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$.

Ans. Let $A=\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

$$\Rightarrow \log A=\lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)\right]^{1/n}+\log\left(1+\frac{3^2}{n^2}\right)+\ldots+\log\left(1+\frac{n^2}{n^2}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{\infty}\log\left(1+\frac{r^2}{n^2}\right)=\int_0^1\log(1+x^2)\,dx$$

$$=\int_0^1\log(1+x^2).1dx=[x\log(1+x^2)]_0^1-\int_0^1\frac{2x.x\,dx}{1+x^2}$$

$$=\log 2-2\int_0^1\frac{(1+x^2)-1}{1+x^2}dx=\log 2-2\int_0^1\left[1-\frac{1}{(1+x^2)}\right]dx$$

$$=\log 2-2[x-\tan^{-1}x]_0^1=\log 2-2\left(1-\frac{\pi}{4}\right)$$

Therefore, $\log A=\log 2+\frac{1}{2}(\pi-4)$

$$\Rightarrow \qquad \log\frac{A}{2}=\frac{1}{2}(\pi-4) \qquad \Rightarrow \qquad A=2e^{(\pi-4)/2}.$$

Q.17. Evaluate $\lim\limits_{n\to\infty} n\left[\frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)(n+4)}+\ldots+\frac{1}{6n^2}\right]$.

Ans. Here, we have

The given limit $=\lim\limits_{n\to\infty} n\sum\limits_{r=1}^{n}\frac{1}{(n+r)(n+2r)}=\lim\limits_{n\to\infty}\frac{n}{n^2}\sum\limits_{r=1}^{n}\frac{1}{(1+r/n)(1+2r/n)}$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\left(1+\frac{r}{n}\right)\left(1+\frac{2r}{n}\right)}=\int_0^1\frac{1}{(1+x)(1+2x)}dx$$

$$=\int_0^1\left[\frac{-1}{1+x}+\frac{2}{1+2x}\right]dx \qquad \text{(Resolving into partial fractions)}$$

$$=[-\log(1+x)+\log(1+2x)]_0^1=\left[\log\frac{(1+2x)}{(1+x)}\right]_0^1$$

$$=\log\frac{3}{2}-\log 1=\log\frac{3}{2}.$$

SECTION-C LONG ANSWER TYPE QUESTIONS

Q.1. Let the function f be defined on $\left[0, \frac{\pi}{4}\right]$ by

$$f(x)=\begin{cases}\cos x, & \text{when } x \text{ is rational}\\ \sin x, & \text{when } x \text{ is irrational}\end{cases}$$

Show that $f \notin R\left[0, \frac{\pi}{4}\right]$.

Ans. Let $P=\left\{\frac{r\pi}{4n} : r=0,1,\ldots,n\right\}$ be any partition, such that $\delta_r=\frac{\pi}{4n}$.

Now, since $\sin x \le \cos x \ \forall\, x \in \left[0, \frac{\pi}{4}\right]$, therefore

$$m_r=\sin(r-1)\frac{\pi}{4n} \quad \text{and} \quad M_r=\cos(r-1)\frac{\pi}{4n}=\sin\left[\frac{\pi}{2}-(r-1)\frac{\pi}{4n}\right]$$

Consider, $U[P,f]-L[P,f]=\sum_{r=1}^{n}[M_r-m_r]\delta_r$

$$=\sum_{r=1}^{n}\left[\sin\left(\frac{\pi}{2}-\frac{(r-1)\pi}{4n}\right)-\sin\frac{(r-1)\pi}{4n}\right]\frac{\pi}{4n}$$

$$=\sum_{r=1}^{n}2\cos\frac{\pi}{4}\sin\left(\frac{\pi}{4}-\frac{(r-1)\pi}{4n}\right)\frac{\pi}{4n}=2.\frac{1}{\sqrt{2}}.\frac{\pi}{4n}\sum_{r=1}^{n}\cos\left(\frac{\pi}{4}-\frac{(r-1)\pi}{4n}\right)$$

$$=\sqrt{2}.\frac{\pi}{4n}[\cos\alpha+\cos(\alpha+\beta)+\ldots+\cos(\alpha+(n-1)\beta)] \text{ with } \alpha=\frac{\pi}{4}, \beta=\frac{\pi}{4n}$$

$$=\frac{\frac{\sqrt{2}.\pi}{4n}}{\sin\frac{\pi}{8n}}\cos\left[\frac{\pi}{4}+\frac{(n-1)\pi}{8n}\right]\sin\frac{n\pi}{8n}=\frac{2\sqrt{2}.\frac{\pi}{8n}}{\sin\frac{\pi}{8n}}\cos\left[\frac{\pi}{4}+\left(1-\frac{1}{n}\right)\frac{\pi}{8}\right]\sin\frac{\pi}{8}$$

$$\Rightarrow \lim_{\|P\|\to 0}[U(P,f)-L(P,f)]=\lim_{n\to\infty}\frac{2\sqrt{2}.\frac{\pi}{8n}}{\sin\frac{\pi}{8n}}\cos\left[\frac{\pi}{4}+\left(1-\frac{1}{n}\right)\frac{\pi}{8}\right]\sin\frac{\pi}{8}$$

$$=2\sqrt{2}\sin\frac{\pi}{8}\cos\left(\frac{\pi}{4}+\frac{\pi}{8}\right)=2\sqrt{2}\sin^2\frac{\pi}{8}=\sqrt{2}\left(1-\cos\frac{\pi}{4}\right)=\sqrt{2}-1\neq 0$$

Hence, $f \notin R\left[0, \frac{\pi}{4}\right]$.

Q.2. Let $f(x)=x^2$ on $[0,a], a>0$. Show that $f\in R[0,a]$. Also, find $\int_0^a f$.

Ans. Let $P=\left[\frac{ra}{n}: r=0,1,\ldots,n\right]$ be any partition of $[0,a]$. Then, clearly, we have

$$m_r=\frac{(r-1)^2\,a^2}{n^2} \text{ and } M_r=\frac{r^2a^2}{n^2}$$

Also,
$$\delta_r=\frac{a}{n}$$

Now,
$$L[P,f]=\sum_{r=1}^{n} m_r\,\delta_r=\sum_{r=1}^{n}\frac{(r-1)^2\,a^2}{n^2}\frac{a}{n}=\frac{a^3}{n^3}\sum_{r=1}^{n}(r-1)^2$$
$$=\frac{a^3}{n^3}\left[\frac{n(n-1)(2n-1)}{6}\right]=\frac{a^3}{6}\left[\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right]$$

and
$$U(P,f)=\sum_{r=1}^{n} m_r\,\delta_r\sum_{r=1}^{n}\frac{r^2a^2}{n^2}\frac{a}{n}$$
$$=\frac{a^3}{n^3}=\sum_{r=1}^{n}r^2=\frac{a^3}{n^3}\frac{n(n+1)(2n+1)}{6}=\frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

Hence,
$$\int_{-0}^{a} f=\lim_{\|P\|\to 0}L(P,f)=\lim_{n\to\infty}\frac{a^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)=\frac{a^3}{3}$$

and
$$\int_{-0}^{a} f=\lim_{\|P\|\to 0}U(P,f)=\lim_{n\to\infty}\frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)=\frac{a^3}{3}$$

Therefore,
$$\int_{-0}^{a} f=\int_{0}^{-a} f$$

which implies $f\in \boldsymbol{R}[0,a]$ and $\int_0^a f=\frac{a^3}{3}$.

Q.3. Show that $f(x)=\sin x$ is integrable on $\left[0,\frac{1}{2}\pi\right]$ and $\int_0^{\pi/2}\sin x\,dx=1$.

Ans. Let $P=\left\{0,\frac{\pi}{2n},\frac{2\pi}{2n},\ldots,\frac{(r-1)\pi}{2n},\frac{r\pi}{2n},\ldots,\frac{n\pi}{2n}=\frac{\pi}{2}\right\}$

be the partition of $[0,\pi/2]$ obtained by dissecting $[0,\frac{1}{2}\pi]$ into n equal parts. The length of each subinterval $=\pi/2n$ and the rth subinterval $I_r=\left[\frac{(r-1)\pi}{2n},\frac{r\pi}{2n}\right]$.

Since $f(x)=\sin x$ is increasing in $[0,\frac{1}{2}\pi]$, we have

$$m_r=\sin\frac{(r-1)\pi}{2n} \text{ and } M_r=\sin\frac{r\pi}{2n}, r=1,2,\ldots,n.$$

Now $U(P,f)=\sum_{r=1}^{n} M_r\,\Delta x_r=\sum_{r=1}^{n}\left(\sin\frac{r\pi}{2n}\right).\frac{\pi}{2n}=\frac{\pi}{2n}\left[\sin\frac{\pi}{2n}+\sin\frac{2\pi}{2n}+\ldots+\sin\frac{n\pi}{2n}\right]$

We know that $\sin a+\sin(a+d)+\ldots+\sin\{a+(n-1)d\}=\dfrac{\sin\left(a+\frac{n-1}{2}d\right)\sin\frac{nd}{2}}{\sin(d/2)}$

$$\therefore\quad U(P,f)=\frac{\pi}{2n}\left[\frac{\sin\left(\frac{\pi}{2n}+\frac{n-1}{2}.\frac{\pi}{2n}\right)\sin\frac{n\pi}{4n}}{\sin\frac{\pi}{4n}}\right]=\frac{\frac{\pi}{2n}\sin\frac{(n+1)\pi}{4n}.\sin\frac{\pi}{4}}{\sin\frac{\pi}{4n}}$$

$$=\frac{\frac{\pi}{2n}.\sin\left(\frac{\pi}{4}+\frac{\pi}{4n}\right).\frac{1}{\sqrt{2}}}{\sin\frac{\pi}{4n}}=\frac{\frac{\pi}{2\sqrt{2n}}\left\{\sin\frac{\pi}{4}\cos\frac{\pi}{4n}+\cos\frac{\pi}{4}\sin\frac{\pi}{4n}\right\}}{\sin\left(\frac{\pi}{4n}\right)}$$

$$=\frac{\pi}{2\sqrt{2n}}.\frac{1}{\sqrt{2}}\left(\cot\frac{\pi}{4n}+1\right)=\frac{\pi}{4n}\left(\cot\frac{\pi}{4n}+1\right).$$

Similarly, we can find that

$$L(P,f)=\frac{\pi}{4n}\left(\cot\frac{\pi}{4n}-1\right).$$

Now $\int_{-0}^{\pi/2} f=\lim_{n\to\infty} L(P,f)=\lim_{n\to\infty}\frac{\pi}{4n}\left(\cot\frac{\pi}{4n}-1\right)$

$$=\lim_{n\to\infty}\frac{(\pi/4n)}{\tan(\pi/4n)}-\lim_{n\to\infty}\frac{\pi}{4n}=1-0=1 \qquad \left[\because \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1\right]$$

and $\int_{0}^{-\pi/2} f=\lim_{n\to\infty} U(P,f)=\lim_{n\to\infty}\frac{\pi}{4n}\left(\cot\frac{\pi}{4n}+1\right)=1.$

Since $\int_{-0}^{\pi/2} f=\int_{0}^{-\pi/2} f,\ f\in R\left[0,\frac{\pi}{2}\right]$ and $\int_{0}^{\pi/2} f=1.$

Q.4. Show that $f(x)=3x+1$ is integrable on $[1,2]$ and $\int_1^2(3x+1)dx=\frac{11}{2}$.

Ans. Here, it is clear that $4\le f(x)\le 7\ \forall\ x\in[1,2]$

$\therefore\ f(x)$ is bounded on [1, 2].

Define a partition $p=\left\{1,1+\frac{1}{n},1+\frac{2}{n},\ldots,1+\frac{n}{n}=2\right\}$ of $[1,2]$. Let I_r be the r^{th} subinterval of P. Then

$$I_r=\left[1+\frac{r-1}{n},1+\frac{r}{n}\right]$$

$$\delta_r=\text{length of } I_r=\frac{1}{n},r=1,2,\ldots,n$$

Let M_r and m_r be respectively the supremum and infimum of f in I_r.

Since $f(x)=3x+1$ is an increasing function in [1, 2], therefore,

$$m_r=3\left(1+\frac{r-1}{n}\right)+1=4+\frac{3(r-1)}{n} \text{ and } M_r=3\left(1+\frac{r}{n}\right)+1=4+\frac{3r}{n}$$

Now $L(P,f)=\sum_{r=1}^{n} m_r\,\delta_r=\sum_{r=1}^{n}\left[\left\{4+\frac{3(r-1)}{n}\right\}.\frac{1}{n}\right]$

$$=\frac{1}{n}\sum_{r=1}^{n}\left\{4+\frac{3(r-1)}{n}\right\}=\frac{1}{n}\left[4n+\frac{3}{n}\sum_{r=1}^{n}(r-1)\right]$$

$$=4+\frac{3}{n^2}[1+2+\ldots+(n-1)]=4+\frac{3}{n^2}.\frac{(n-1)n}{2}=4+\frac{3}{2}\left(1-\frac{1}{n}\right)$$

$$\Rightarrow \int_{-1}^{2} f(x)\,dx=\lim_{n\to\infty} L(P,f)=4+\frac{3}{2}=\frac{11}{2}$$

Also, $U(P,f)=\sum_{r=1}^{n} M_r\delta_r=\sum_{r=1}^{n}\left[\left(1+\frac{3r}{n}\right)\frac{1}{n}\right]=\frac{1}{n}\sum_{r=1}^{n}\left(4+\frac{3r}{n}\right)=\frac{1}{n}\left[4n+\frac{3}{n}\sum_{r=1}^{n} r\right]$

$$=4+\frac{3}{n}(1+2+\ldots+n)=4+\frac{3}{n}.\frac{n(n+1)}{2}=4+\frac{3}{2}\left(1+\frac{1}{n}\right)$$

$$\int_{1}^{-2} f(x)\,dx=\lim_{n\to\infty} U(P,f)=4+\frac{3}{2}=\frac{11}{2}$$

Clearly $\int_{1}^{-2} f(x)\,dx=\int_{1}^{-2} f(x)\,dx$

Hence, $f(x)$ is ***R***-integrable on [1, 2] and $\int_{1}^{2} f(x)\,dx=\frac{11}{2}$.

Q.5. If $f(x)=\cos x,\ \forall\, x\in[0,\pi/2]$. Show that f is integrable on $[0,\pi/2]$ and $\int_{0}^{\pi/2}\cos x\,dx=1$.

Ans. Since $0\le f(x)\le 1\ \forall\ x\in[0,\pi/2]$

Therefore, $f(x)=\cos x$ is bounded on $[0,\pi/2]$.

Define a position $P=\left\{0,\frac{\pi}{2n},\frac{2\pi}{2n},\ldots,\frac{r\pi}{2n},\ldots,\frac{n\pi}{2n}=\frac{\pi}{2}\right\}$ of $\left[0,\frac{\pi}{2}\right]$. Let $I_r=\left[\frac{(r-1)\pi}{2n},\frac{r\pi}{2n}\right]$ be the r^{th} subinterval of the partition P, with length $\delta_r=\frac{\pi}{2n}, r=1,2,\ldots,n.$

Now let M_r and m_r be supremum and infimum respectively of f in I_r.

We know that $f(x)=\cos x$ is a decreasing function in $[0,\pi/2]$, therefore

$$M_r=\cos\frac{(r-1)\pi}{2n} \text{ and } m_r=\cos\frac{r\pi}{2n}$$

$$\therefore \quad U(P,f)=\sum_{r=1}^{n} M_r\,\delta_r=\sum_{r=1}^{n}\left[\left\{\cos\frac{(r-1)\pi}{2n}\right\}\frac{\pi}{2n}\right]$$

$$=\frac{\pi}{2n}\left[\cos\theta+\cos\frac{\pi}{2n}+\cos\frac{2\pi}{2n}+\ldots+\cos\frac{(n-1)\pi}{2n}\right]$$

$$=\frac{\pi}{2n}\cdot\frac{\cos\theta\left(+\frac{(n-1)}{2}\cdot\frac{\pi}{2n}\right).\sin\frac{n\pi}{4n}}{\sin(\pi/4n)}$$

$$\left[\cos A+\cos(A+B)+\ldots+\cos(A+(n-1)B)=\frac{\cos\left(A+\frac{(n-1)}{2}B\right)\sin\frac{nB}{2}}{\sin(B/2)}\right]$$

$$=\frac{(\pi/4n)}{\sin(\pi/4n)}2\cos\left\{\frac{\pi}{4}\left(1-\frac{1}{n}\right)\right\}\sin\frac{\pi}{4}$$

$$\Rightarrow \quad \int_0^{-\pi/2} f(x)\,dx=\lim_{n\to\infty} U(P,f)=1.2\cos\frac{\pi}{4}\sin\frac{\pi}{4}=1$$

Also, $L(P,f)=\sum_{r=1}^{n} m_r\,\delta_r=\sum_{r=1}^{n}\left[\left(\cos\frac{r\pi}{2n}\right).\frac{\pi}{2n}\right]=\frac{\pi}{2n}\left[\cos\frac{\pi}{2n}+\cos\frac{2\pi}{2n}+\ldots+\cos\frac{n\pi}{2n}\right]$

$$=\frac{\frac{\pi}{2n}\left[\cos\left\{\frac{\pi}{2n}+\frac{(n-1)}{2}\cdot\frac{\pi}{2n}\right\}\sin\frac{n\pi}{4n}\right]}{\sin(\pi/4n)}=\frac{(\pi/4n)}{\sin(\pi/4n)}2\cos\left\{\frac{\pi}{4}\left(1+\frac{1}{n}\right)\right\}\sin\frac{\pi}{4}$$

$$\Rightarrow \quad \int_{-0}^{\pi/2} f(x)\,dx=\lim_{n\to\infty} L(P,f)=1.2\cos\frac{\pi}{4}\sin\frac{\pi}{4}=1.$$

Therefore $\int_0^{\pi/2} f(z)\,dx=\int_0^{\pi/2} f(x)=dx=1$

Hence, $f(x)=\cos x$ is $\boldsymbol{R}$ integrable on $[0,\pi/2]$.

Also, $\int_0^{\pi/2} f(x)\,dx=\int_0^{\pi/2}\cos x\,dx=1.$

Q.6. Write and prove Bonnet's mean value theorem.

Ans. Theorem (Bonnet's Mean Value Theorem) : Let $g\in \boldsymbol{R}[a,b]$ and let f be monotonic and non-negative on $[a,b]$. Then for some ξ or $\eta\in[a,b]$

$$\int_a^b f(x)g(x)\,dx=f(a)\int_a^{\xi} g(x)\,dx \quad \text{or} \quad \int_a^b f(x)g(x)\,dx=f(b)\int_{\eta}^{b} g(x)\,dx$$

according as f is monotonically non-increasing or non-decreasing on $[a,b]$.

Proof : If $a=b$, the result is trivial. Let $b>a$ and let f be non-negative and monotonically non-increasing on $[a,b]$.

Let $P=\{a=x_0,x_1,\ldots,x_n=b\}$

be any partition of $[a,b]$. Let m_r, M_r be the bounds of g on $[x_{r-1}, x_r]$ and ξ_r any point on $[x_{r-1}, x_r]$. Then

$$m_r \Delta x_r \le g(\xi_r)\Delta x_r \le M_r \Delta x_r$$

and $$m_r \Delta x_r \le \int_{x_{r-1}}^{x_r} g \le M_r \Delta x_r.$$ [Theorem 1 of article 8]

On summing for each $r = 1, 2, \dots, p \le n$, we get

$$\sum_1^p m_r \Delta x_r \le \sum_1^p g(\xi_r)\Delta x_r \le \sum_1^p M_r \Delta x_r \quad \text{...(1)}$$

and $$\sum_1^p m_r \Delta x_r \le \int_a^{x_p} g \le \sum_1^p M_r \Delta x_r. \quad \text{...(2)}$$

Then (1) and (2) give

$$\left| \int_a^{x_p} g - \sum_1^p g(\xi_r)\Delta x_r \right| \le \sum_1^p (M_r - m_r)\Delta x_r$$

$$\le \sum_1^n (M_r - m_r)\Delta x_r = U(P,g) - L(P,g)$$

$$= \omega(P,g) \quad \text{...(3)}$$

[Note that if $b > a$, the inequalities (1) and (2) are reversed but (3) remains the same].

Now by theorem 1 of article 9, $\int_a^x g$ is continuous on $[a,b]$ and hence is bounded on $[a,b]$.

Let *m, M* be its bounds on $[a,b]$. Then (3) gives

$$m - \omega(P,f) \le \sum_1^p g(\xi_r)\Delta x_r \le M + \omega(P,f).$$

Using Abel's lemma*, we get

$$f(a)[m - \omega(P,f)] \le \sum_1^p f(\xi_r) g(\xi_r)\Delta x_r$$

$$\le f(a)[M - \omega(P,f)]. \quad \text{...(4)}$$

Since f is monotonic, we have $f \in \mathbf{R}[a,b]$.

Also $\quad g \in \mathbf{R}[a,b]$.

Hence $\quad fg \in \mathbf{R}[a,b]$.

Now $\quad f \in \mathbf{R}[a,b] \Rightarrow \omega(P,f) \to 0$ as $\|P\| \to 0$.

Q.7. Evaluate $\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}dx$.

Ans. Let $$u = \int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}dx. \quad \text{...(1)}$$

Here the integrand $\frac{\tan^{-1}(ax)}{x(1+x^2)}$ is a function of two variables x and a. Obviously u is a function of a. So differentiating both sides of (1) with respect to a, we get

$$\frac{du}{da}=\frac{d}{da}\left[\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}dx\right]=\int_0^\infty\left[\frac{\partial}{\partial a}\left\{\frac{\tan^{-1}(ax)}{x(1+x^2)}\right\}\right]dx$$

$$=\int_0^\infty \frac{1}{x(1+x^2)}\cdot\frac{1}{1+a^2x^2}.x\,dx=\int_0^\infty \frac{dx}{(1+x^2)(1+a^2x^2)}$$

$$=\int_0^\infty\left[\frac{1}{(1-a^2)(1+x^2)}-\frac{a^2}{(1-a^2)(1+a^2x^2)}\right]dx,$$

resolving the integrand into partial fractions

$$=\frac{1}{(1-a^2)}\Big[\tan^{-1}x\Big]_0^\infty-\frac{a^2}{1-a^2}\cdot\frac{1}{a}\Big[\tan^{-1}ax\Big]_0^\infty$$

$$=\frac{1}{(1-a^2)}\cdot\frac{\pi}{2}-\frac{a}{1-a^2}\left[\frac{\pi}{2}\right]$$ [Assuming that a is positive]

$$=\frac{\pi}{2}\cdot\frac{1}{1-a^2}(1-a)=\frac{\pi}{2(1+a)}.\text{ Thus }\frac{du}{da}=\frac{\pi}{2(1+a)}$$

Integrating both sides with respect to a, we get

$$u=\frac{\pi}{2}\log(1+a)+C. \qquad ...(2)$$

When $a=0$, we have from (1) $u=0$.

$\therefore$ from (2), $0=0+C$ or $C=0$.

Putting $C=0$ in (2), we get $u=\frac{1}{2}\pi\log(1+a)$.

Hence $$\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}dx=\frac{\pi}{2}\log(1+a),\text{ if } a>0$$

Case when a is negative :

If a is negative, we have

$$\frac{du}{da}=\frac{1}{(1-a^2)}[\tan^{-1}x]_0^\infty-\frac{a^2}{1-a^2}\cdot\frac{1}{a}[\tan^{-1}ax]_0^\infty$$

$$=\frac{1}{(1-a)^2}\left[\frac{\pi}{2}-a\left(-\frac{\pi}{2}\right)\right] \qquad \left[\because \tan^{-1}(-\infty)=-\frac{\pi}{2}\right]$$

$$=\frac{\pi}{2(1-a)}.$$

Integrating both sides w.r.t. a, we get $u=-\frac{\pi}{2}\log(1-a)+C_1$. ...(3)

Again, when $a=0, u=0$. Putting these values in (3), we have $C_1=0$.

$$\therefore \qquad u=-\frac{\pi}{2}\log(1-a).$$

Hence $$\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}dx=-\frac{\pi}{2}\log(1-a),\text{ if } a<0.$$

Q.8. Evaluate $\int_0^\infty \frac{\cos mx}{1+x^2}\,dx, m>0$ and deduce the value of $\int_0^\infty \frac{\sin mx}{1+x^2}\,dx$.

Ans. Let $u=\int_0^\infty \frac{\cos mx}{1+x^2}\,dx.$

Then $\frac{du}{dm}=-\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx=-\int_0^\infty \frac{x^2\sin mx}{x(1+x^2)}\,dx=-\int_0^\infty \frac{\{(1+x^2)-1\}\sin mx}{x(1+x^2)}\,dx$

$$=-\int_0^\infty \frac{\sin mx}{x}\,dx+\int_0^\infty \frac{\sin mx}{x(1+x^2)}\,dx=-\frac{\pi}{2}+\int_0^\infty \frac{\sin mx}{x(1+x^2)}\,dx. \quad ...(2)$$

Differentiating both sides of (2) w.r.t. m, we get

$$\frac{d^2u}{dm^2}=\int_0^\infty \frac{x\cos mx}{x(1+x^2)}\,dx=\int_0^\infty \frac{\cos mx}{1+x^2}\,dx=u$$

or $(D^2-1)u=0$, where $D\equiv\frac{d}{dm}$.

The general solution of this differential equation is $u=Ae^m+Be^{-m}$.

$\therefore$ $\frac{du}{dm}=Ae^m-Be^{-m}$.

Now when $m=0$, from (1) we have $u=\int_0^\infty \frac{dx}{1+x^2}=[\tan^{-1}x]_0^\infty=\frac{\pi}{2}$

and from (2), $\frac{du}{dm}=-\frac{\pi}{2}$.

So from (3) and (4), we get $\frac{\pi}{2}=A+B$ and $-\frac{\pi}{2}=A-B$.

Solving these equations for A and B, we get $A=0$ and $B=\pi/2$.

Putting these values of A and B in (3), we get

$$u=\frac{\pi}{2}e^{-m}. \quad \text{Hence} \quad \int_0^\infty \frac{\cos mx}{1+x^2}\,dx=\frac{\pi}{2}e^{-m}.$$

Differentiating both sides w.r.t. m, we get

$$-\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx=-\frac{\pi}{2}e^{-m} \quad \text{or} \quad \int_0^\infty \frac{x\sin mx}{1+x^2}\,dx=\frac{\pi}{2}e^{-m}.$$

Q.9. Show that $\int_0^{\pi/2}\log(\alpha^2\cos^2\theta+\beta^2\sin^2\theta)\,d\theta=\pi\log\left(\frac{\alpha+\beta}{2}\right)$.

Ans. Let $u=\int_0^{\pi/2}\log(\alpha^2\cos^2\theta+\beta^2\sin^2\theta)\,d\theta.$ ...(1)

Then $\frac{du}{d\alpha}=\int_0^{\pi/2}\frac{2\alpha\cos^2\theta}{\alpha^2\cos^2\theta+\beta^2\sin^2\theta}\,d\theta=\int_0^{\pi/2}\frac{2\alpha\cos^2\theta}{(\alpha^2-\beta^2)\cos^2\theta+\beta^2}\,d\theta$

$$= \frac{2\alpha}{(\alpha^2 - \beta^2)} \int_0^{\pi/2} \frac{(\alpha^2 - \beta^2)\cos^2\theta + \beta^2 - \beta^2}{(\alpha^2 - \beta^2)\cos^2\theta + \beta^2} d\theta$$

$$= \frac{2\alpha}{(\alpha^2 - \beta^2)} \int_0^{\pi/2} \left[1 - \frac{\beta^2}{(\alpha^2 - \beta^2)\cos^2\theta + \beta^2}\right] d\theta$$

$$= \frac{2\alpha}{(\alpha^2 - \beta^2)} \int_0^{\pi/2} \left[1 - \frac{\beta^2 \sec^2\theta}{(\alpha^2 - \beta^2) + \beta^2 \sec^2\theta}\right] d\theta$$

$$= \frac{2\alpha}{(\alpha^2 - \beta^2)} \int_0^{\pi/2} \left[1 - \frac{\beta^2 \sec^2\theta}{\alpha^2 + \beta^2 \tan^2\theta}\right] d\theta$$

$$= \frac{2\alpha}{(\alpha^2 - \beta^2)} \left[\theta - \beta . \frac{1}{\alpha} \tan^{-1}\left(\frac{\beta \tan\theta}{\alpha}\right)\right]_0^{\pi/2}$$

[Putting $\beta \tan\theta = t$ so that $\beta \sec^2\theta \, d\theta = dt$]

$$= \frac{2\alpha}{\alpha^2 - \beta^2} \left[\frac{\pi}{2} - \frac{\beta}{\alpha} . \frac{\pi}{2}\right] = \frac{\pi}{\alpha + \beta}. \text{ Thus } \frac{du}{d\alpha} = \frac{\pi}{\alpha + \beta}.$$

Integrating both sides with respect to α, we get

$$u = \pi \log(\alpha + \beta) + C. \qquad ...(2)$$

From (1), when $\alpha = \beta$, we have

$$u = \int_0^{\pi/2} \log\{\alpha^2 (\cos^2\theta + \sin^2\theta)\} d\theta = \int_0^{\pi/2} \log\alpha^2 \, d\theta$$

$$= \frac{\pi}{2} \log\alpha^2 = \pi \log\alpha.$$

So putting $\beta = \alpha$ in (2), we get

$$\pi \log\alpha = \pi \log(2\alpha) + C \quad \text{or} \quad C = \pi \log\frac{1}{2}.$$

Hence putting $C = \pi \log\frac{1}{2}$ in (2), we get

$$u = \pi \log(\alpha + \beta) + \pi \log\frac{1}{2} = \pi \log\left(\frac{\alpha + \beta}{2}\right).$$

❑

UNIT VI

SECTION-A (VERY SHORT ANSWER TYPE) QUESTIONS

Q.1. Define improper integrals.

Ans. The definite integral $\int_a^b f(x)\,dx$ is said to be an improper integral if (i) the interval (a,b) is not finite (*i.e.,* is infinite) and the function $f(x)$ is bounded over this interval; or (ii) the interval (a,b) is finite and $f(x)$ is not bounded over this interval; or (iii) neither the interval (a,b) is finite nor $f(x)$ is bounded over it.

Q.2. Define convergence of improper integral.

Ans. Convergence of Improper Integral : The integral $\int_a^\infty f(x)\,dx$ is said to converge to the value I, if for any arbitrary chosen positive number ε, however small but not zero, there exists a positive number N such that $\left|\int_a^b f(x)\,dx - I\right| < \varepsilon$; for all values of $b \geq N$.

If the integral $f(x)$ has a finite limit then improper integral called convergent and if having no finite limit *i.e.,* limits are $+\infty, -\infty$ then it is said to be divergent and when having neither finite value $0, +\infty$ nor $-\infty$, then improper integrals is said to be oscillatory.

Q.3. Test the convergence of (i) $\int_{-\infty}^0 e^x\,dx$; (ii) $\int_{-\infty}^0 e^{-x}\,dx$.

Ans. (i) We have $\int_{-\infty}^0 e^x\,dx = \lim_{x\to\infty} \int_{-x}^0 e^x\,dx$, (By def.)

$$= \lim_{x\to\infty} [e^x]_{-x}^0 = \lim_{x\to\infty} [1 - e^{-x}] = [1-0] = 1.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

(ii) We have $\int_{-\infty}^0 e^{-x}\,dx = \lim_{x\to\infty} \int_{-x}^0 e^{-x}\,dx$, (By def.)

$$= \lim_{x\to\infty} \left[\frac{e^{-x}}{-1}\right]_{-x}^0 = -\lim_{x\to\infty} [e^0 - e^x] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent (*i.e.,* the integral does not exist).

Q.4. Test the convergence of $\int_0^\infty e^{-mx}\,dx, (m > 0)$.

Ans. We have $\int_0^\infty e^{-mx}\,dx = \lim_{x\to\infty} \int_0^\infty e^{-mx}\,dx$, (by def.)

$$= \lim_{x\to\infty} \left[\frac{e^{-mx}}{-m}\right]_0^x = \lim_{x\to\infty} \left\{-\frac{1}{m}(e^{-mx} - 1)\right\} = -\frac{1}{m}[0-1] = \frac{1}{m}.$$

Thus the limit exists and is unique and finite, therefore the given integral is convergent.

Q.5. Test the convergence of $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Ans. We have $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{x\to\infty} \int_{-x}^{0} \frac{dx}{1+x^2} + \lim_{x\to\infty} \int_{0}^{x} \frac{dx}{1+x^2}$

$$= \lim_{x\to\infty} [\tan^{-1} x]_{-x}^{0} + \lim_{x\to\infty} [\tan^{-1} x]_{0}^{x}$$

$$= \lim_{x\to\infty} [0 - \tan^{-1}(-x)] + \lim_{x\to\infty} [\tan^{-1} x - 0]$$

$$= -(-\pi/2) + \pi/2 = \pi.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

Q.6. Evaluate $\int_{-1}^{1} \frac{dx}{x^2}$.

Ans. Here the integrand becomes infinite at $x = 0$ and $-1 < 0 < 1$.

$$\therefore \quad \int_{-1}^{1} \frac{dx}{x^2} = \lim_{\varepsilon\to 0} \int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \lim_{\varepsilon'\to 0} \int_{\varepsilon'}^{1} \frac{dx}{x^2}$$

$$= \lim_{\varepsilon\to 0} \left[-\frac{1}{x}\right]_{-1}^{-\varepsilon} + \lim_{\varepsilon'\to 0} \left[-\frac{1}{x}\right]_{\varepsilon'}^{1}$$

$$= \lim_{\varepsilon\to 0} \left[\frac{1}{\varepsilon} - 1\right] + \lim_{\varepsilon'\to 0} \left[-1 + \frac{1}{\varepsilon'}\right]$$

Since both the limits do not exist finitely, therefore the integral does not exist and is divergent.

Q.7. Evaluate $\int_{0}^{1} \frac{dx}{\sqrt{x}}$.

Ans. In the given integral, the integrand $1/\sqrt{x}$ becomes infinite at the lower limit $x = 0$. Therefore we have

$$\int_{0}^{1} \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0} \int_{0+\varepsilon}^{1} \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0} [2\sqrt{x}]_{\varepsilon}^{1} = \lim_{\varepsilon\to 0} [2 - 2\sqrt{\varepsilon}] = 2.$$

Hence the given integral is convergent and its value is 2.

Q.8. Discuss the convergence of the integral $\int_{0}^{1} \frac{dx}{\sqrt{1-x}}$ by evaluating.

Ans. Here given integral is $\int_{0}^{1} \frac{dx}{\sqrt{1-x}}$.

It is not bounded at limit $x = 1$.

So $\quad \int_{0}^{1} \frac{dx}{\sqrt{1-x}} = \lim_{\varepsilon\to 0} \int_{0}^{1-\varepsilon} \frac{dx}{\sqrt{1-x}} = \lim_{\varepsilon\to 0} [-2\sqrt{1-x}]_{0}^{1-\varepsilon} = \lim_{\varepsilon\to 0} [-2\sqrt{\varepsilon} + 2] = 2.$

which is finite a number.

$\Rightarrow$ the given integral is convergent.

Q.9. Discuss the convergence of the integral $\int_1^\infty \frac{dx}{x^{3/2}}$ by evaluating.

Ans. Since we have $\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{X\to\infty} \int_1^X x^{-3/2}\, dx = \lim_{X\to\infty} \left[\frac{x^{-1/2}}{-\frac{1}{2}}\right]_1^X$

$$= \lim_{X\to\infty} \left[-\frac{2}{\sqrt{x}}\right]_1^X = \lim_{X\to\infty} \left[-\frac{2}{\sqrt{X}}+2\right] = \frac{-2}{\infty}+2 = 2.$$

⇒ the integral exist and finite.

⇒ the given integral is convergent.

Q.10. Test the convergence of $\int_a^\infty (1-e^{-x})\frac{\cos x}{x^2}\, dx$, when $a>0$.

Ans. Let $f(x)=\frac{\cos x}{x^2}$ and $\phi(x)=1-e^{-x}$.

We have $\left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$ as $|\cos x|\le 1$.

Since $\int_a^\infty \frac{1}{x^2}\, dx$ is convergent, therefore by comparison test $\int_a^\infty \frac{\cos x}{x^2}\, dx$ is also convergent.

Again $\phi(x)=1-e^{-x}$ is monotonic increasing and bounded function for $x>a$.

Hence by Abel's test $\int_a^\infty (1-e^{-x})\frac{\cos x}{x^2}\, dx$ is convergent.

Q.11. Show that $\int_0^1 x^{n-1} e^{-x}\, dx$ is convergent if $n>0$.

Ans. If $n\ge 1$, then $\int_0^1 x^{n-1} e^{-x}\, dx$ is a proper integral because the integrand $f(x)=x^{n-1} e^{-x}$ is bounded in the interval $(0,1)$. So the given integral is convergent when $n\ge 1$.

If $0<n<1$, the integrand $f(x)=x^{n-1} e^{-x}$ is unbounded at $x=0$. Take $g(x)=x^{n-1}$.

Then $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} e^{-x} = 1$, which is finite and non-zero.

∴ by comparison test, $\int_0^1 f(x)\, dx$ and $\int_0^1 g(x)\, dx$ either both converge or both diverge.

But $\int_0^1 g(x)\, dx = \int_0^1 x^{n-1}\, dx = \lim_{\varepsilon\to 0} \int_\varepsilon^1 x^{n-1}\, dx = \lim_{\varepsilon\to 0} \left[\frac{x^n}{n}\right]_\varepsilon^1$

$$= \lim_{\varepsilon\to 0} \left[\frac{1}{n}-\frac{\varepsilon^n}{n}\right] = \frac{1}{n}, \text{ which is a definite real number.}$$

∴ $\int_0^1 g(x)\, dx$ is convergent. Hence $\int_0^1 x^{n-1} e^{-x}\, dx$ is also convergent.

Q.12. Show that the integral $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ is convergent.

Ans. In the given integral, the integrand $f(x)=\frac{1}{x^{1/3}(1+x^2)}$ is umbounded at the lower limit of integration $x=0$. Take $g(x)=1/x^{1/3}$.

Then $\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1}{1+x^2}=1$, which is finite and non-zero.

$\therefore$ by comparison test $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$

either both converge or both diverge. But the comparison integral $\int_0^1 \frac{dx}{x^{1/3}}$ is convergent because here $n=1/3$ which is less than 1. Hence the integral $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ is also convergent.

Q.13. Examine the convergence of $\int_1^\infty \frac{dx}{x^{1/3}(1+x^{1/2})}$.

Ans. Let $f(x)=\frac{1}{x^{1/3}(1+x^{1/2})}=\frac{1}{x^{1/3}.x^{1/2}(1+1/x^{1/2})}=\frac{1}{x^{5/6}.\{1+(1/x^{1/2})\}}$

$f(x)$ is bounded in the interval $(1,\infty)$ then by μ-test $\mu=\frac{5}{6}-0=\frac{5}{6}$.

We have $\lim_{x\to\infty} x^{\mu} f(x)=\lim_{x\to\infty} x^{5/6}.\frac{1}{x^{5/6}\{1+1/x^{1/2}\}}$

$=\lim_{x\to\infty}\frac{1}{(1+1/x^{1/2})}=1$ (finite and non-zero)

Since $\mu=5/6<1$, so the given integral is divergent.

Q.14. Examine the convergence of $\int_0^\infty \frac{x\,dx}{(1+x)^3}$.

Ans. We have $\int_0^\infty \frac{x\,dx}{(1+x)^3}=\int_0^a \frac{x\,dx}{(1+x)^3}+\int_0^\infty \frac{x\,dx}{(1+x)^3}$

$\int_0^\infty \frac{x\,dx}{(1+x)^3}$ is convergent because it is a proper integral. Also, the integrand $\frac{x}{(1+x)^3}$ is bounded throughout the finite interval $]0,a[$, we need to check the convergence of $\int_a^\infty \frac{x\,dx}{(1+x)^3}$.

Let $f(x)=\frac{x}{(1+x)^3}$ then $f(x)$ is bounded in the interval $]a,\infty[$. Take $\mu=3-1=2$.

then $$\lim_{x\to\infty} x^{\mu} f(x) = \lim_{x\to\infty} x^2 \cdot \frac{x}{(1+x)^3} = \lim_{x\to\infty} \frac{1}{\{1+(1/x)\}^3} = 1.$$

Since $\mu = 2$, *i.e.*, >1, therefore by μ-test the integral $\int_a^{\infty} \frac{x\,dx}{(1+x)^3}$ is convergent.

Q.15. Show that the integral $\int_0^{\pi/2} \log \sin x\, dx$ converges.

Ans. The only point of infinite discontinuity of the integrand is $x = 0$.

Now $\lim_{x\to 0} x^{\mu} \log \sin x$, when $\mu > 0 = \lim_{x\to 0} \frac{\log \sin x}{x^{-\mu}}$, $\left[\text{from } \frac{\infty}{\infty}\right]$

$$= \lim_{x\to 0} \frac{\cot x}{-\mu x^{-\mu-1}} = \lim_{x\to 0} -\frac{1}{\mu} \cdot \frac{x^{\mu+1}}{\tan x} \qquad \left[\text{from } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} -\frac{1}{\mu} \cdot \frac{(\mu+1) x^{\mu}}{\sec^2 x}. \qquad \text{[by L'Hospital's rule]}$$

$= 0$, if $\mu > 0$.

Taking μ between 0 and 1, it follows from μ-test that the given integral is convergent.

Q.16. Test the convergence of the integral $\int_0^1 \frac{dx}{x^3(1-x^2)}$.

Ans. Here, it is clear that the integral $f(x) = \frac{1}{x^3(1+x^2)}$ is unbounded at $x = 0$.

Let $$\phi(x) = \frac{1}{x^3}$$

$\therefore$ $$\lim_{x\to 0} \frac{f(x)}{\phi(x)} = \lim_{x\to 0} \frac{1}{1+x^2} = 1,$$ *i.e.*, finite and non-zero.

Then, by comparison test $\int_0^1 f(x)\,dx$ and $\int_0^1 \phi(x)\,dx$ either both converges or both diverges.

But clearly $\int_0^1 \frac{dx}{x^3}$ is convergent.

Hence, the given integral $\int_0^1 \frac{dx}{x^3(1+x^2)}$ is convergent.

Q.17. Show that $\int_0^{\infty} \frac{x^c}{c^x}\,dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}, c > 0.$

Ans. $$\int_0^{\infty} \frac{x^c}{c^x}\,dx = \int_0^{\infty} x^c c^{-x}\,dx = \int_0^{\infty} x^c \,[e^{\log_e c}]^{-x}\,dx \qquad [\because c = e^{\log_e c} \text{ if } c \ge 0]$$

$$= \int_0^\infty x^{(c+1)-1} e^{-x \log_e c}\, dx = \frac{\Gamma(c+1)}{(\log_e c)^{c+1}}$$

$$\left[\because \int_0^\infty x^{n-1} e^{-kx}\, dx = \frac{\Gamma(n)}{k^n} \text{ where } n>0, k>0\right]$$

Q.18. Evaluate $\int_0^\infty t^{-3/2}(1-e^{-t})dt$.

Ans. $$\int_0^\infty t^{-3/2}(1-e^{-t})\,dt = (1-e^{-t})\left[\frac{t^{-1/2}}{-1/2}\right]_0^\infty - \int_0^\infty (e^{-t})\left(\frac{t^{-1/2}}{-1/2}\right)dt$$

$$= 0 + 2\int_0^\infty e^{-t}\, t^{(1/2)-t}\, dt$$

$$= 2\Gamma(1/2) \qquad \text{[By definition of gamma function]}$$

$$= 2\sqrt{\pi}.$$

Q.19. Prove that $\int_0^\infty \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}} dx = 0, m>0, n>0.$

Ans. The given integral is $= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx - \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$

$$= \mathrm{B}(m,n) - \mathrm{B}(n,m) = \mathrm{B}(m,n) - \mathrm{B}(m,n) = 0.$$

SECTION-B SHORT ANSWER TYPE QUESTIONS

Q.1. Evaluate $\int_1^\infty \frac{x+2}{x(x+1)} dx.$

Ans. Here, the upper limit of the integral is infinite. So it is first kind of improper integral. Now

$$\int_1^\infty \frac{x+2}{x(x+1)} dx = \lim_{b\to\infty} \int_1^b \frac{x+2}{x(x+1)} dx$$

$$= \lim_{b\to\infty} \int_1^b \left(\frac{2}{x} - \frac{1}{x+1}\right) dx \qquad \text{(By resolving into partial fraction)}$$

$$= \lim_{b\to\infty}\left[\log \frac{x^2}{x+1}\right]_1^b = \lim_{b\to\infty}\left[\log \frac{b^2}{b+1} + \log 2\right]$$

$$= \lim_{b\to\infty}\left[\log \frac{1}{\frac{1}{b}+\frac{1}{b^2}} + \log 2\right] = \log\infty + \log 2 = \infty + \log 2 = \infty.$$

Here, the limit of integral is infinite so the given integral is divergent.

Q.2. Discuss the convergence of the following integrals by evaluating them

(a) $\int_1^\infty \frac{dx}{\sqrt{x}}$, **(b)** $\int_1^\infty \frac{dx}{x^{3/2}}$.

Ans. (i) We have $\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{x\to\infty}\int_1^x \frac{dx}{\sqrt{x}}$, (By def.)

$$= \lim_{x\to\infty}\int_1^x x^{-1/2}\,dx = \lim_{x\to\infty}\left[\frac{x^{1/2}}{1/2}\right]_1^x = \lim_{x\to\infty}[2\sqrt{x}-2] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent (*i.e.,* the integral does not exist).

(ii) We have $\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{x\to\infty}\int_1^x \frac{dx}{x^{3/2}}$, (By def.)

$$= \lim_{x\to\infty}\int_1^x x^{-3/2}\,dx = \lim_{x\to\infty}\left[\frac{x^{-1/2}}{-1/2}\right]_1^x = \lim_{x\to\infty}\left[-\frac{2}{\sqrt{x}}\right]_1^x$$

$$= \lim_{x\to\infty}\left[-\frac{2}{\sqrt{x}}+2\right] = 2.$$

Thus the limit exist and is unique and finite; therefore the given integral is convergent and its value is 2.

Q.3. Test the convergence of the integral $\int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx$.

Ans. Here $f(x) = \frac{\cos mx}{x^2+a^2}$. Let $g(x) = \frac{1}{x^2+a^2}$

Obviously $g(x)$ is positive in the interval $(0,\infty)$.

We have $|f(x)| = \left|\frac{\cos mx}{x^2+a^2}\right| = \frac{|\cos mx|}{x^2+a^2} \le \frac{1}{x^2+a^2}$, since $|\cos mx| \le 1$.

Thus $|f(x)| \le g(x)$ when $x \ge 0$.

$\therefore$ by comparison test, $\int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx$ is convergent if $\int_0^\infty \frac{dx}{x^2+a^2}$ is convergent.

But $\int_0^\infty \frac{dx}{x^2+a^2} = \lim_{x\to\infty}\int_0^\infty \frac{dx}{x^2+a^2} = \lim_{x\to\infty}\left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^x$

$$= \lim_{x\to\infty}\left[\frac{1}{a}\tan^{-1}\frac{x}{a} - 0\right] = \frac{1}{a}.\frac{\pi}{2} = \text{a definite real number.}$$

$\therefore$ $\int_0^\infty \frac{dx}{x^2+a^2}$ is convergent. Hence $\int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx$ is also convergent.

Q.4. Test the convergence of the integral $\int_0^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx$, where m and n are positive integers.

Ans. Let $a > 0$. We have

$$\int_0^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx = \int_0^{a} \frac{x^{2m}}{1+x^{2n}}\, dx + \int_0^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx.$$

The first integral on the right hand side is a proper integral and so it is convergent. Therefore the given integral is convergent or divergent according as $\int_0^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx$ is convergent or divergent.

To test the convergence of $\int_a^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx$, let us take $\mu = 2n-2m$.

We have $\lim\limits_{x\to\infty} x^{\mu} \cdot \frac{x^{2m}}{1+x^{2n}} = \lim\limits_{x\to\infty} x^{2n-2m} \cdot \frac{x^{2m}}{x^{2n}\{1+(1/x^{2n})\}} = \lim\limits_{x\to\infty} \frac{1}{1+(1/x^{2n})} = 1,$

which is finite and non-zero.

$\therefore$ by μ-test, the given integral is convergent if $\mu > 1$ *i.e.,* if $2n - 2m > 1$ which is possible if $n > m$ since m and n are positive integers. Also by μ-test, the given integral is divergent if $\mu \le 1$ *i.e.,* if $2n - 2m \le 1$ *i.e.,* if $n \le m$ since n and m are positive integers.

Q.5. Show that $\int_0^{\infty} \frac{\sin mx}{a^2+x^2}\, dx$ converges absolutely.

Ans. If $\int_0^{\infty} \left| \frac{\sin mx}{a^2+x^2} \right| dx$ is convergent, then the integral $\int_0^{\infty} \frac{\sin mx}{a^2+x^2}\, dx$ will be absolutely convergence.

Let $f(x) = \left| \frac{\sin mx}{a^2+x^2} \right|$ then $f(x)$ is bounded in the interval $]0, \infty[$, we have

$$f(x) = \left| \frac{\sin mx}{a^2+x^2} \right| = \frac{|\sin mx|}{a^2+x^2} \le \frac{1}{a^2+x^2}, \text{ since } |\sin mx| \le 1.$$

$\therefore$ By comparison test, $\int_0^{\infty} f(x)\, dx$ is convergent if $\int_0^{\infty} \frac{1}{a^2+x^2}\, dx$ is convergent.

But $\int_0^{\infty} \frac{dx}{a^2+x^2} = \lim\limits_{x\to\infty} \int_0^{x} \frac{dx}{a^2+x^2}$

$$= \lim_{x\to\infty} \left[\frac{1}{a} \tan^{-1} \frac{x}{a} \right]_0^x = \lim_{x\to\infty} \left[\frac{1}{a} \tan^{-1} \frac{x}{a} - 0 \right] = \frac{1}{\alpha} \cdot \frac{\pi}{2}$$

which is a definite real number.

$\therefore \int_0^\infty \frac{dx}{a^2+x^2}$ is convergent. Hence $\int_0^\infty f(x)\,dx$ is also convergent and so the given integral is absolutely convergent.

Q.6. Test the convergence of the integral $\int_0^{\pi/2} \frac{\cos x}{x^2}\,dx$.

Ans. Here, the integral $f(x)=\frac{\cos x}{x^2}$ is unbounded at $x=0$.

Let $$\phi(x)=\frac{1}{x^2}.$$

Then $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\left\{\frac{\cos x}{x^2}.x^2\right\}$$

$$=\lim_{x\to\infty}\cos x=1, \text{ finite and non-zero.}$$

$\therefore$ By comparison test the integrals $\int_0^{\pi/2} f(x)\,dx$ and $\int_0^{\pi/2}\phi(x)\,dx$, either both converge or both diverge.

But $$\int_0^{\pi/2}\phi(x)\,dx=\int_0^{\pi/2}\frac{1}{x^2}\,dx=\lim_{h\to 0}\int_h^{\pi/2}\frac{1}{x^2}\,dx$$

$$=\lim_{h\to 0}\left[-\frac{1}{x}\right]_h^{\pi/2}=\lim_{h\to 0}\left[-\frac{2}{\pi}+\frac{1}{h}\right]=\infty.$$

$\therefore \int_0^{\pi/2}\phi(x)\,dx$ is divergent.

Hence, the integral $\int_0^{\pi/2}\frac{\cos x}{x^2}\,dx$ is divergent.

Q.7. Discuss the convergence of the integral $\int_0^\infty \frac{\sin mx}{a^2+x^2}\,dx$ and show that it will be absolutely convergent.

Ans. Since given that $I=\int_0^\infty \frac{\sin mx}{a^2+x^2}\,dx$ will be absolutely convergent.

If $\int_0^\infty \left|\frac{\sin mx}{a^2+x^2}\right| dx$ is convergent.

(Since every absolutely convergent function is convergent)

Let $f(x)=\left|\frac{\sin mx}{a^2+x^2}\right|$ then $f(x)$ is bounded in the interval $(0,\infty)$.

$$f(x)=\frac{|\sin mx|}{a^2+x^2}\le\frac{1}{a^2+x^2}$$

Now we use comparison test

$\therefore \int_0^\infty f(x)\,dx$ will be convergent if $\int_0^\infty \frac{1}{a^2+x^2}$ is convergent.

$$\int_0^\infty \frac{dx}{a^2+x^2} = \lim_{x\to\infty}\int_0^x \frac{dx}{a^2+x^2} = \lim_{x\to\infty}\left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_a^x = \lim_{x\to\infty}\left[\frac{1}{a}\tan^{-1}\frac{x}{a}-0\right] = \frac{\pi}{2a}$$

$\Rightarrow$ a finite and real number.

$\Rightarrow$ convergent.

Hence, $\int_0^\infty f(x)\,dx$ will be convergent.

Q.8. Prove that $\int_a^\infty \frac{\cos\alpha x - \cos\beta x}{x}\,dx$ is convergent where $a > 0$.

Ans. We have $\int_a^\infty \frac{\cos\alpha x - \cos\beta x}{x}\,dx = \int_a^\infty \frac{\cos\alpha x}{x}\,dx - \int_a^\infty \frac{\cos\beta x}{x}\,dx$

The function $f(x) = 1/x$ is bounded and monotonically decreasing for all $x \geq a$ and $\lim_{x\to\infty}\frac{1}{x} = 0$.

Now, $$\left|\int_a^x \cos\alpha x\,dx\right| = \left|\frac{1}{\alpha}(\sin\alpha x - \sin\alpha a)\right| \leq \frac{2}{|\alpha|}$$

$\therefore \left|\int_a^x \cos\alpha x\,dx\right|$ is bounded for all finite values of x.

Similarly $\left|\int_a^x \cos\beta x\,dx\right|$ is bounded for all finite values of x.

By Dirichlet's test both the integral $\int_a^\infty \frac{\cos\alpha x}{x}\,dx$ and $\int_a^\infty \frac{\cos\beta x}{x}\,dx$ are convergent.

Hence, the given integral (being the difference of two convergent integral) also convergent.

Q.9. Discuss the convergence of the integral $\int_0^1 x^{n-1}\log x\,dx$.

Ans. (i) Since $\lim_{x\to 0} x^r \log x = 0$ where $r > 0$, the integral is a proper integral, when $n > 1$.

(ii) When $n = 1$, we have

$$\int_0^1 \log x\,dx = \lim_{\varepsilon\to 0}\int_\varepsilon^1 \log x\,dx = \lim_{\varepsilon\to 0}[x\log x - x]_\varepsilon^1$$

$$= \lim_{\varepsilon\to 0}[-1 - \varepsilon\log\varepsilon + \varepsilon] = -1.$$

$\therefore$ the integral is convergent if $n = 1$.

(iii) Let $n < 1$ and $f(x) = x^{n-1}\log x$.

Then $$\lim_{x\to 0} x^\mu f(x) = \lim_{x\to 0} x^{\mu+n-1}\log x$$

$$= 0 \quad \text{if} \quad \mu > 1 - n \qquad \text{...(1)}$$

and
$$= -\infty \quad \text{if} \quad \mu \le 1 - n. \qquad \text{...(2)}$$

Hence when $0 < n < 1$, we can choose μ between 0 and 1 and satisfying (1). The integral is therefore convergent by μ-test when $0 < n < 1$.

Again when $n \le 0$, we can take $\mu = 1$ and satisfying (2). Hence by μ-test the integral is divergent when $n \le 0$.

Therefore from (i), (ii) and (iii), we conclude that the given integral is convergent when $n > 0$ and divergent when $n \le 0$.

Q.10. With certain limitation on the values of *a, b, m* and *n*, prove that

$$\int_0^\infty \int_0^\infty e^{-(ax^2+by^2)}\, x^{2m-1}\, y^{2n-1}\, dx\, dy = \frac{\Gamma(m)\Gamma(n)}{4a^m b^n}.$$

Ans. Let
$$I = \int_0^\infty \int_0^\infty e^{-(ax^2+by^2)}\, x^{2m-1}\, y^{2n-1}\, dx\, dy \qquad \text{...(1)}$$

$$I = \int_0^\infty e^{-ax^2}\, x^{2m-1}\, dx \times \int_0^\infty e^{-by^2}\, y^{2n-1}\, dy = I_1 \times I_2 \qquad \text{...(2)}$$

where
$$I_1 = \int_0^\infty e^{-ax^2}\, x^{2m-1}\, dx \qquad \text{...(3)}$$

$$I_2 = \int_0^\infty e^{-by^2}\, y^{2n-1}\, dy. \qquad \text{...(4)}$$

Put $ax^2 = t, x = (t/a)^{1/2}$ so that $dx = dt/2\sqrt{at}$ then equation (3) becomes

$$I_1 = \int_0^\infty e^{-t} \left[\frac{t}{a}\right]^{\frac{(2m-1)}{2}} \frac{dt}{2\sqrt{at}} = \frac{1}{2a^m} \int_0^\infty e^{-t}\, t^{m-1}\, dt$$

$$= \frac{\Gamma(m)}{2a^m}$$ [By definition of gamma function taking $n > 0, a > 0$]

Then $I_2 = \dfrac{\Gamma(n)}{2b^n}$ if $n > 0, b > 0$.

$\therefore$ From (1) and (2), we get

$$I = I_1 \times I_2 = \frac{\Gamma(m)\Gamma(n)}{4a^m b^n}.$$

Q.11. Show that $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\, dx = \frac{1}{(a+b)^m \cdot a^n} B(m,n).$

Ans. The given integral

$$I = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\, dx = \int_0^1 \left(\frac{x}{a+bx}\right)^{m-1} \cdot \left(\frac{1-x}{a+bx}\right)^{n-1} \cdot \frac{1}{(a+bx)^2}\, dx.$$

Put $\dfrac{x}{a+bx} = \dfrac{y}{a+b}$ so that $\dfrac{(a+bx).1 - x.b}{(a+bx)^2}\, dx = \dfrac{dy}{a+b}$

i.e., $$\frac{1}{(a+bx)^2}\,dx=\frac{dy}{a\,(a+b)}$$

Further $$\frac{1-x}{a+bx}=\frac{1}{a}\,\frac{a-ax}{a+bx}=\frac{1}{a}\left[\frac{a+bx-ax-bx}{a+bx}\right]=\frac{1}{a}\left[1-\frac{x\,(a+b)}{a+bx}\right]=\frac{1-y}{a}.$$

Also when $x=0, y=0$ and when $x=1, y=1$.

$$\therefore \qquad I=\int_0^1\left(\frac{y}{a+b}\right)^{m-1}\left(\frac{1-y}{a}\right)^{n-1}.\frac{dy}{a\,(a+b)}$$

$$=\frac{1}{(a+b)^m\,.\,a^n}\int_0^1 y^{m-1}\,(1-y)^{n-1}\,dy=\frac{\mathbf{B}\,(m,n)}{(a+b)^m\,.\,a^n}$$

Q.12. Show that if m, n are positive, then

$$\int_a^b (x-a)^{m-1}\,(b-x)^{n-1}\,dx=(b-a)^{m+n-1}\,.\,\mathbf{B}(m,n)$$

$$=(b-a)^{m+n-1}\,.\,\frac{\Gamma m\,\Gamma n}{\Gamma\,(m+n)}$$

Ans. The given integral is $\int_a^1 (x-a)^{m-1}\,(b-x)^{n-1}\,dx.$

Put $x=a+(b-a)\,y$ so that $dx=(b-a)\,dy$.

Also when $x=a, y=0$ and when $x=b, y=1$.

$$\therefore \quad \int_a^b (x-a)^{m-1}\,(b-x)^{n-1}\,dx$$

$$=\int_0^1 [(b-a)\,y]^{m-1}\,[b-a-(b-a)\,y]^{n-1}\,.\,(b-a)\,dy$$

$$=\int_0^1 (b-a)^{m-1}\,.\,y^{m-1}\,.\,(b-a)^{n-1}\,.\,(1-y)^{n-1}\,.\,(b-a)\,dy$$

$$=(b-a)^{m+n-1}\int_0^1 y^{m-1}\,(1-y)^{n-1}\,dy=(b-a)^{m+n-1}\,\mathbf{B}\,(m,n)$$

$$=(b-a)^{m+n-1}\,.\,\frac{\Gamma m\,\Gamma n}{\Gamma\,(m+n)}. \qquad \left[\because \mathbf{B}\,(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma\,(m+n)}\right]$$

Q.13. Prove that $\int_0^2 (8-x^3)^{-1/3}\,dx=\dfrac{2\pi}{3\sqrt{3}}$.

Ans. Let $x^3=8t$, then $x=2t^{1/3}$

$$dx=\frac{2}{3}\,t^{-2/3}\,dt,$$

when $x=0$, to $x=2, t=0$ to $t=1$

$$\int_0^2 (8-x^3)^{-1/3}\,dx=\int_0^1 (8-8t)^{-1/3}\,.\,\frac{2}{3}\,t^{-2/3}\,dt$$

$$=(8)^{-1/3}\,.\,\frac{2}{3}\int_0^1 t^{-2/3}\,(1-t)^{-1/3}\,dt$$

$$=\frac{1}{3}\int_0^1 t^{(1/3)-t}\,(1-t)^{(2/3)-1}\,dt=\frac{1}{3}B\left(\frac{1}{3},\frac{2}{3}\right)=\frac{1}{3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)}$$

$$=\frac{1}{3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)}{\Gamma(1)}=\frac{1}{3}\frac{\pi}{\sin\frac{\pi}{3}}=\frac{2\pi}{3\sqrt{3}}.$$

Q.14. Show that $B(n,n+1)=\dfrac{1}{2}\dfrac{\Gamma(n)^2}{\Gamma(2n)}$ and hence deduce that

$$\int_0^{\pi/2}\left(\frac{1}{\sin^3\theta}-\frac{1}{\sin^2\theta}\right)^{1/4}\cos\theta\,d\theta=\frac{\{\Gamma(1/4\}^2}{2\sqrt{\pi}}.$$

Ans. We have $B(n,n+1)=\dfrac{\Gamma(n)\Gamma(n+1)}{\Gamma(n+n+1)}=\dfrac{\Gamma(n).n\Gamma n}{(2n)\Gamma(2n)}$ ...(1)

$$B(n,n+1)=\frac{1}{2}\frac{\{\Gamma(n)\}^2}{\Gamma(2n)} \qquad [\because \Gamma(p+1)=p\Gamma(p)]$$

Let $I=\displaystyle\int_0^{\pi/2}\left(\frac{1}{\sin^3\theta}-\frac{1}{\sin^2\theta}\right)^{1/4}\cos\theta\,d\theta.$ ...(2)

Putting $x=\sin\theta$, so that $dx=\cos\theta\,d\theta$ in (2), we get

$$I=\int_0^1\left(\frac{1}{x^3}-\frac{1}{x^2}\right)^{1/4}dx=\int_0^1\left(\frac{1-x}{x^3}\right)^{1/4}dx$$

$$=\int_0^1 x^{-3/4}(1-x)^{1/4}\,dx=\int_0^1 x^{(1/4)-1}(1-x)^{(5/4)-1}\,dx$$

$$=B\left(\frac{1}{4},\frac{5}{4}\right),\text{ by definition of beta function}$$

$$=B\left(\frac{1}{4},\frac{1}{4}+1\right)=\frac{1}{2}\frac{\{\Gamma(1/4)\}^2}{\Gamma(1/2)}.(n=1/4)=\frac{\{\Gamma(1/4)\}^2}{2\sqrt{\pi}}.$$

Q.15. Show that $I=\displaystyle\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta.\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}}=\pi.$

Ans. We know that

$$\int_0^{\pi/2}\sin^p d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\sqrt{\pi}}{2\Gamma\left(\frac{p+2}{2}\right)} \qquad ...(1)$$

$$I=\int_0^{\pi/2}\sin^{1/2}\theta\,.\,d\theta\,.\int_0^{\pi/2}\sin^{-1/2}\theta\,d\theta=\frac{\Gamma\left(\frac{1/2+1}{2}\right)\sqrt{\pi}\,.\,\Gamma\left(\frac{-1/2+1}{2}\right)\sqrt{\pi}}{2\Gamma\left(\frac{1/2+2}{2}\right).2\Gamma\left(\frac{-1/2+2}{2}\right)}$$

$$=\frac{\Gamma(3/4)\sqrt{\pi}}{2\Gamma(5/4)}\cdot\frac{\Gamma(1/4)\sqrt{\pi}}{2\Gamma(3/4)}=\frac{\pi\Gamma(1/4)}{4\Gamma(1+1/4)}=\frac{\pi\Gamma(1/4)}{4\,.\,1/4\,.\,\Gamma(1/4)}=\pi.$$

Q.16. Evaluate $\int_0^1\frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$.

Ans. We have $\int_0^1\frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx=\int_0^1\frac{x^{m-1}\,dx}{(1+x)^{m+n}}+\int_0^1\frac{x^{n-1}\,dx}{(1+x)^{m+n}}$. ...(1)

Now in the second integral on the R.H.S. of (1), we put $x=1/y$ so that $dx=-(1/y^2)\,dy$; also when $x\to 0$, $y\to\infty$ and when $x=1$, $y=1$.

$$\therefore\quad \int_0^1\frac{x^{n-1}\,dx}{(1+x)^{m+n}}=\int_\infty^1\frac{(1/y)^{n-1}}{(1+1/y)^{m+n}}\left(-\frac{1}{y^2}dy\right)$$

$$=-\int_\infty^1\frac{y^{m+n}\,dy}{(1+y)^{m+n}\,.\,y^{n-1}\,.\,y^2}=\int_1^\infty\frac{y^{m-1}}{(1+y)^{m+n}}dy$$

$$=\int_1^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx.\qquad\left[\because\int_a^b f(x)\,dx=\int_a^b f(y)\,dy\right]$$

Now from (1), we have

$$\int_0^1\frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx=\int_0^1\frac{x^{m-1}}{(1+x)^{m+n}}dx+\int_1^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$=\int_0^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx,\text{ by a property of definite integrals}$$

$$=\mathbf{B}(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Q.17. Show that $\int_0^\infty\cos(bz^{1/n})dz=\frac{1}{b^n}\Gamma(n+1).\cos\frac{n\pi}{2}$.

Ans. Put $z^{1/n}=x$ *i.e.,* $z=x^n$, so that $dz=nx^{n-1}\,dx$.

$$\therefore\quad\int_0^\infty\cos(bz^{1/n})\,dz=\int_0^\infty\cos(bx).nx^{n-1}\,dx=n\int_0^\infty x^{n-1}\cos(bx)\,dx$$

$$=\text{real part of } n\int_0^\infty e^{-ibx}\,x^{n-1}\,dx$$

$$=\text{real part of } n\frac{\Gamma(n)}{(ib)^n}$$

$$= \text{real part of } \frac{n\Gamma(n)}{b^n}.(\cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi)^{-n}$$

$$= \text{real part of } \frac{\Gamma(n+1)}{b^n}\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$$

$$= \frac{1}{b^n}.\Gamma(n+1).\cos\left(\frac{n\pi}{2}\right).$$

Q.18. Express $\Gamma(1/6)$ in terms of $\Gamma(1/3)$.

Ans. By duplication formula, we have

$$\Gamma(n)\Gamma(n+1/2) = \frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n). \quad ...(1)$$

Putting $n = 1/6$ in (1), we get

$$\Gamma(1/6)\Gamma(2/3) = \frac{\sqrt{\pi}\,\Gamma(1/3)}{2^{-2/3}} \Rightarrow \Gamma(1/6) = \frac{\sqrt{\pi}\,\Gamma(1/3)}{2^{-2/3}\,\Gamma(2/3)}. \quad ...(2)$$

Also, we know that

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}. \quad ...(3)$$

Putting $n = 1/3$ in (3), we get

$$\Gamma(1/3)\Gamma(2/3) = \frac{\pi}{\sin(\pi/3)} = 2\pi/\sqrt{3}$$

$$\Gamma(2/3) = \frac{2\pi}{\sqrt{3}\,\Gamma(1/3)}. \quad ...(4)$$

Substituting the value of $\Gamma(2/3)$ given by (4) in (2), we get

$$\Gamma(1/6) = \frac{\sqrt{\pi}\,\Gamma(1/3)}{2^{-2/3}}.\frac{\sqrt{3}\,\Gamma(1/3)}{2\pi} = \frac{\sqrt{3}}{2^{1/3}\sqrt{\pi}}[\Gamma(1/3)]^2$$

SECTION-C LONG ANSWER TYPE QUESTIONS

Q.1. Test the convergence of $\int_0^\infty e^{-x}\frac{\sin x}{x}dx$.

Ans. We can write $\int_0^\infty e^{-x}\frac{\sin x}{x}dx = \int_0^1 e^{-x}\frac{\sin x}{x}dx + \int_1^\infty e^{-x}\frac{\sin x}{x}dx.$

Since $\lim_{x\to 0} e^{-x}\frac{\sin x}{x} = 1$, therefore the integrand $e^{-x}\frac{\sin x}{x}$ is bounded throughout the finite interval (0, 1). So $\int_0^1 e^{-x}\frac{\sin x}{x}dx$ is a proper integral and therefore it is convergent.

Thus we need to check the convergence of $\int_1^\infty e^{-x}\frac{\sin x}{x}dx$ only.

Let $f(x)=e^{-x}\dfrac{\sin x}{x}$. Then $f(x)$ is bounded in the interval $(1,\infty)$.

Take $g(x)=e^{-x}$. Then $g(x)$ is positive in the interval $(1,\infty)$.

We have $$|f(x)|=\left|e^{-x}\frac{\sin x}{x}\right|=e^{-x}.|\sin x|.\frac{1}{x}$$

$$\le e^{-x},\text{ since } |\sin x|\le 1 \text{ and } \frac{1}{x}\le 1.$$

Thus $|f(x)|\le g(x)$ throughout the interval $(1,\infty)$.

$\therefore$ by comparison test $\int_1^\infty f(x)\,dx$ is convergent if $\int_1^\infty g(x)\,dx$ is convergent.

Now $$\int_1^\infty g(x)\,dx=\int_1^\infty e^{-x}\,dx=\lim_{x\to\infty}\int_1^\infty e^{-x}\,dx=\lim_{x\to\infty}[-e^{-x}]_1^x$$

$$=\lim_{x\to\infty}[-e^{-x}+e^{-1}]=0+e^{-1}=1/e,$$

which is a definite finite number. Hence $\int_1^\infty g(x)\,dx$ is convergent.

$\therefore$ $\int_1^\infty f(x)\,dx$ is also convergent. Hence $\int_0^\infty e^{-x}\dfrac{\sin x}{x}\,dx$ is convergent because the sum of two convergent integrals is also convergent.

Q.2. Show that the integral $\int_0^\infty e^{-x^2}\,dx$ is convergent.

Ans. We have $\int_0^\infty e^{-x^2}\,dx=\int_0^1 e^{-x^2}\,dx+\int_1^\infty e^{-x^2}\,dx.$

Obviously $\int_0^1 e^{-x^2}\,dx$ is a proper integral because here the interval of integration (0, 1) is finite and the integrand e^{-x^2} is bounded throughout this interval. Therefore this integral is convergent. So we need to check the convergence of $\int_1^\infty e^{-x^2}\,dx$ only.

Let $f(x)=e^{-x^2}$. Take $g(x)=xe^{-x^2}$ so that $g(x)$ is positive throughout the interval $(1,\infty)$. We have $|f(x)|=e^{-x^2}\le xe^{-x^2}$, since $x\ge 1$.

Thus $|f(x)|\le g(x)$ throughout the interval $(1,\infty)$.

$\therefore$ by comparison test $\int_1^\infty e^{-x^2}\,dx$ is convergent if $\int_1^\infty xe^{-x^2}\,dx$ is convergent.

Now $$\int_1^\infty xe^{-x^2}\,dx=\lim_{x\to\infty}\int_1^x xe^{-x^2}dx=\lim_{x\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)_1^x$$

$$=\lim_{x\to\infty}\left(-\frac{1}{2}e^{-x^2}+\frac{1}{2}e^{-1}\right)$$

$$=\frac{1}{2}e^{-1},\text{ which is a definite number.}$$

$\therefore \int_1^\infty xe^{-x^2} dx$ is convergent and so $\int_1^\infty e^{-x^2} dx$ is also convergent.

Hence the given integral $\int_1^\infty e^{-x^2} dx$ is also convergent as it is the sum of two convergent integrals.

Q.3. Discuss the convergence of the given integral $\int_0^\infty x^{n-1} e^{-x} dx$, if $n > 0$.

Ans. Here given that $I = \int_0^\infty x^{n-1} e^{-x} dx$

$$I = \int_0^1 x^{n-1} e^{-x} dx + \int_0^\infty x^{n-1} e^{-x} dx$$

Let $I_1 = \int_0^1 x^{n-1} e^{-x} dx$

$$I_2 = \int_1^\infty x^{n-1} e^{-x}.$$

Here for discuss the convergence of given integral, we use μ-test in I_2 and comparison test in I_1.

$$I_1 = \int_0^1 x^{n-1} e^{-x} dx$$

$f(x) = x^{n-1} e^{-x}$ at $x = 0$, will be unbounded.

Let $g(x) = x^{n-1}$

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} e^{-x} = 1.$$

By comparison test if $g(x)$ is convergent then $f(x)$ will also be convergent of if divergent then $f(x)$ will be divergent.

$$\int_0^1 g(x) dx = \int_0^1 x^{n-1} dx = \lim_{\varepsilon \to 0} \int_\varepsilon^1 x^{n-1} dx$$

$$= \lim_{\varepsilon \to 0} \left[\frac{x^n}{n} \right]_\varepsilon^1 = \lim_{\varepsilon \to 0} \left[\frac{1}{n} - \frac{\varepsilon^n}{n} \right]$$

$= \frac{1}{n}$, which is a finite real number.

$\Rightarrow \int_0^1 g(x) dx$ is convergent.

$\Rightarrow f(x)$ will be convergent.

Now $I_2 = \int_0^\infty x^{n-1} e^{-x} dx$

How, $f(x) = x^{n-1} e^{-x}$. It is bounded in the interval $(1, \infty)$.

$$\lim_{x \to 0} x^\mu f(x) = \lim_{x \to 0} \frac{x^\mu . x^{n-1}}{e^x} = \lim_{x \to \infty} \frac{x^{\mu+n-1}}{1 + x + \frac{x^2}{2!}} = 0.$$

For $\mu > 1$, we have $\int_1^{\infty} x^{n-1} e^{-x}\, dx$ is convergent.

From the above result we can say I will be convergent because I_1 and I_2 both are convergent.

Q.4. Discuss the convergence or divergence of the integral $\int_0^{\infty} \frac{x^{a-1}}{1+x} dx$.

Ans. Let $f(x) = \frac{x^{a-1}}{1+x}$. If $b > 0$, we can write

$$\int_0^{\infty} \frac{x^{a-1}}{1+x} dx = \int_0^{b} \frac{x^{a-1}}{1+x} dx + \int_b^{\infty} \frac{x^{a-1}}{1+x} dx = I_1 + I_2, \text{ say.}$$

Let $a \geq 1$. Then $f(x)$ is bounded throughout the interval $(0, b)$ and so the integral I_1 is a proper integral and hence it is convergent. To test the convergence of the infinite integral I_2 in this case, we have

$$\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} x^{\mu} \cdot \frac{x^{a-1}}{1+x} = \lim_{x \to \infty} \frac{x^{\mu + a - 1}}{x+1}$$

$$= 1, \text{ if } \mu + a - 1 = 1 \text{ } i.e., \text{ if } \mu = 2 - a \text{ which is } \leq 1 \text{ since } a \geq 1.$$

Hence by μ-test I_2 is divergent.

$\therefore$ the given integral is divergent if $a \geq 1$.

Let $a < 1$. Then in the interval $(0, b)$, $f(x)$ is unbounded only at $x = 0$. Also $f(x)$ is bounded throughout the interval (b, ∞). Therefore in this case I_1 is an improper integral of the second kind and I_2 is an improper integral of the first kind. To test the convergence of I_1, we have

$$\lim_{x \to 0} x^{\mu} \cdot \frac{x^{a-1}}{x+1} = \lim_{x \to 0} \frac{x^{\mu + a - 1}}{x+1} = 1, \text{ if } \mu + a - 1 = 0 \text{ } i.e., \text{ if } \mu = 1 - a.$$

If we take $0 < a < 1$, then we have $0 < \mu < 1$ and so by μ-test I_1 is convergent. If we take $a \leq 0$, then $\mu \geq 1$ and so by μ-test I_1 is divergent.

To test the convergence of I_2 when $a < 1$, we have

$$\lim_{x \to \infty} x^{\mu} \cdot \frac{x^{a-1}}{x+1} = \lim_{x \to \infty} \frac{x^{\mu + a - 1}}{x+1} = 1, \text{ if } \mu + a - 1 = 1 \text{ } i.e., \text{ if } \mu = 2 - a \text{ which is } > 1 \text{ since } a < 1.$$

Hence by μ-test I_2 is convergent if $a < 1$.

Thus I_2 is convergent if $a < 1$. But I_1 is convergent if $0 < a < 1$ and is divergent if $a \leq 0$.

$\therefore$ the given integral is convergent if $0 < a < 1$ and is divergent if $a \leq 0$.

Hence the given integral is convergent if $0 < a < 1$ and is divergent if $a \geq 1$ or if $a \leq 0$.

Q.5. Discuss the convergence of the Beta function $\int_0^1 x^{m-1} (1-x)^{n-1} dx$.

Ans. Let $f(x) = x^{m-1} (1-x)^{n-1}$.

The following different cases arise :

(i) When m and n are both ≥ 1, the integrand $f(x)$ is bounded throughout the interval $(0, 1)$ and so the given integral is a proper integral and is convergent.

(ii) When m and n are both <1, the integrand $f(x)$ becomes infinite both at $x=0$ and at $x=1$. In this case we take $0<a<1$ and we write

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^a x^{m-1}(1-x)^{n-1}\,dx + \int_a^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= I_1 + I_2, \text{ say.}$$

In the case of the integral I_1, the interval of integration is $(0, a)$ and so the integrand is unbounded at $x=0$ only. To test the convergence of I_1, we have

$$\lim_{x\to 0} x^{\mu}\,.\,f(x) = \lim_{x\to 0} x^{\mu}\,.\,x^{m-1}(1-x)^{n-1}$$

$$= \lim_{x\to 0} x^{\mu+m-1}(1-x)^{n-1}$$

$$= 1, \text{ if } \mu + m - 1 = 0 \text{ i.e., if } \mu = 1 - m.$$

If we take $0<m<1$, we have $0<\mu<1$ and so by μ-test I_1 is convergent. If we take $m \le 0$, we have $\mu \ge 1$ and so by μ-test I_1 is divergent.

Again in the case of the integral I_2, the interval of integration is $(a, 1)$ and so the integrand is unbounded at $x=1$ only. To test the convergence of I_2, we have

$$\lim_{x\to 1-0}(1-x)^{\mu}\,.\,f(x) = \lim_{x\to 1-0}(1-x)^{\mu}\,x^{m-1}(1-x)^{n-1}$$

$$= \lim_{x\to 1-0}(1-x)^{\mu+n-1}\,x^{m-1}$$

$$= \lim_{\varepsilon\to 0}\{1-(1-\varepsilon)\}^{\mu+n-1}(1-\varepsilon)^{m-1}$$

$$= \lim_{\varepsilon\to 0}\varepsilon^{\mu+n-1}(1-\varepsilon)^{m-1}$$

$$= 1, \text{ if } \mu + n - 1 = 0 \text{ i.e., if } \mu = 1 - n.$$

If we take $0<n<1$, we have $0<\mu<1$ and so by μ-test I_2 is convergent. If we take $n \le 0$, we have $\mu \ge 1$ and so by μ-test I_2 is divergent.

Thus if m and n are both <1, the given integral is convergent only if $0<m<1$ and $0<n<1$.

(iii) When $m<1$ and $n \ge 1$, the integrand $f(x)$ is unbounded only at $x=0$. In this case by μ-test, the given integral is convergent if $0<m<1$ and is divergent if $m \le 0$.

Again if $m \ge 1$ and $n<1$, the integrand $f(x)$ is unbounded only at $x=1$. In this case by μ-test, the given integral is convergent if $0<n<1$ and is divergent if $n \le 0$.

Hence from (i), (ii) and (iii) it follows that the given integral is convergent if both m and n are >0 and divergent otherwise.

Q.6. Evaluate the following integrals

(i) $\int_0^1 x^4(1-x)^2\,dx$ **(ii)** $\int_0^a y^4\sqrt{a^2-y^2}\,dy$

(iii) $\int_0^2 x(8-x^3)^{1/3}\,dx$ **(iv)** $\int_0^\infty \frac{x\,dx}{1+x^6}$.

Ans. (i) We have

$$\int_0^1 x^4(1-x)^2\,dx = \int_0^1 x^{5-1}(1-x)^{3-1}\,dx$$

$$=\frac{\Gamma(5)\Gamma(3)}{\Gamma(5+3)}=\frac{4!2!}{7!}=\frac{4!\times 2}{7\times 5\times 4!\times 6}=\frac{1}{105}.$$

(ii) $\int_0^a y^4\sqrt{a^2-y^2}\,dy.$

Ist Method. Let $y^2=a^2t$ so that $dy=\frac{a^2dt}{2y}=\frac{a\,dt}{2\sqrt{t}}$, then

$$I=\int_0^1 (a^2t)^2\sqrt{(a^2-ta^2)}\left(\frac{a\,dt}{2\sqrt{t}}\right)$$

$$=\frac{a^6}{2}\int_0^1 t^{3/2}(1-t)^{1/2}\,dt$$

$$=\frac{a^6}{2}\int_0^1 t^{(5/2)-1}(1-t)^{(3/2)-1}\,dt$$

$$=\frac{a^6}{2}\frac{\Gamma(5/2)\Gamma(3/2)}{\Gamma(5/2+3/2)}=\frac{a^6}{2}\frac{3/2.1/2\sqrt{\pi}.1/2\sqrt{\pi}}{3!}=\frac{\pi a^6}{32}$$

IInd Method. Let $y=a\sin\theta$, so that $dy=a\cos\theta\,d\theta$, then

$$I=\int_0^{\pi/2} a^4.\sin^4\theta\times a\cos\theta\times a\cos\theta\,d\theta$$

$$=\frac{a^6\,\Gamma 5/2\,\Gamma 3/2}{2\Gamma\left(\frac{4+2+2}{2}\right)}=a^6\frac{3/2.1/2\sqrt{\pi}\times 1/2\sqrt{\pi}}{2\times 3!}=\pi a^6/32$$

(iii) Let $\int_0^2 x(8-x^3)^{1/3}\,dx=I$

Put $x^3=8t$ or $x=2t^{1/3}$ so that $dx=\frac{2}{3}t^{-2/3}\,dt.$

$$\therefore\quad I=\int_0^1 (2t^{1/3})(8-8t)^{1/3}(2/3\,t^{-2/3}\,dt)$$

$$=\frac{8}{3}\frac{\Gamma(2/3)\Gamma(4/3)}{\Gamma(2/3+4/3)}=\frac{8}{3}\frac{\Gamma(1-1/3)\Gamma(1+1/3)}{\Gamma(2)}$$

$$=\frac{8}{3}\Gamma(1-1/3).\frac{1}{3}\Gamma(1/3)=\frac{8}{9}\frac{\pi}{\sin\pi/3}=\frac{16\pi}{2\sqrt{3}}.\qquad\left(\because\ \begin{array}{c}\Gamma 2=1!=1\\ \Gamma(1+p=p\Gamma p)\\ \Gamma(1-n)\Gamma n=\dfrac{\pi}{\sin n\pi}\end{array}\right)$$

(iv) Let $\quad I=\int_0^\infty \frac{x\,dx}{1+x^6}.$

Put $\quad x^6=y$ or $x=y^{1/6}.$

So that $dx = \frac{1}{6} . y^{-5/6}\, dy$

$$I = \frac{1}{6}\int_0^\infty \frac{y^{1/6} . y^{-5/6}}{1+y}\, dy = \frac{1}{6}\int_0^\infty \frac{y^{-2/3}}{1+y}\, dy$$

$$= \frac{1}{6}\int_0^\infty \frac{y^{(1/3)-1}}{(1+y)^{2/3+1/3}}\, dy = \frac{1}{6} B\,(1/3, 2/3)$$

$$= \frac{1}{6}\frac{\Gamma(1/3)\Gamma(2/3)}{\Gamma(1/3+2/3)} = \frac{1}{6}\frac{\Gamma(1/3)\Gamma(1-1/3)}{\Gamma 1} = \frac{1}{6}\frac{\pi}{\sin\frac{\pi}{3}}$$

$$\left[\because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{1}{6} . \frac{\pi}{(\sqrt{3}/2)} = \frac{1}{6} . \frac{2\pi}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}}.$$

Q.7. (a) $\int_0^\pi \frac{\sin^{n-1} x\, dx}{(a+b\cos x)^n} = \frac{2^{n-1}}{(a^2-b^2)^{n/2}} B(n/2, n/2).$

(b) $\int_0^\pi \frac{\sqrt{\sin x}}{[5+3\cos x]^{3/2}} = \frac{[\Gamma(3/4)]^2}{2\sqrt{2}\,\pi}.$

Ans. (a) Let $I = \int_0^\pi \frac{\sin n-1\, x\, dx}{(a+b\cos x)^n} = \int_0^\pi \frac{(\sin x)^{n-1}\, dx}{(a+b\cos x)^n}$ ...(1)

$$\therefore \quad I = \int_0^\pi \frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^{n-1} dx}{\left[a\left\{\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)\right\}+b\left\{\cos^2\left(\frac{x}{2}\right)-\sin^2\frac{x}{2}\right\}\right]^n} \quad \text{[By (1)]}$$

$$= 2^{n-1}\int_0^\pi \frac{\sin^{n-1}\left(\frac{x}{2}\right)\cos^{n-1}\left(\frac{x}{2}\right)dx}{\left[(a+b)\cos^2\left(\frac{x}{2}\right)+(a-b)\sin^2\left(\frac{x}{2}\right)\right]^n}$$

$$= \frac{2^{n-1}}{(a+b)^n}\int_0^\pi \frac{\sin^{n-1}\left(\frac{x}{2}\right)\cos^{n-1}\left(\frac{x}{2}\right)dx}{\cos^{2n}\left(\frac{x}{2}\right)\left[1+\frac{a-b}{a+b}\tan^2\frac{x}{2}\right]^n}$$

$$= \frac{2^{n-1}}{(a+b)^n} \int_0^\infty \frac{\tan^{n-1} \cdot \left(\frac{x}{2}\right) \sec^2 \left(\frac{x}{2}\right) dx}{\left[1 + \frac{a-b}{a+b} \tan^2 \frac{x}{2}\right]^n}$$

$$= \frac{2^{n-1}}{(a+b)^n} \int_0^\infty \frac{\left[\frac{a+b}{a-b} t\right]^{\frac{(n-2)}{2}} \cdot \frac{a+b}{a-b} dt}{(1+t)^n}$$

$$\left[\text{Put } \frac{a-b}{a+b} \tan^2 \frac{x}{2} = t, \therefore 2 \frac{a-b}{a+b} \tan \frac{x}{2} \sec^2 \frac{x}{2} \cdot \frac{dx}{2} = dt\right]$$

$$= \frac{2^{n-1}}{[(a+b)(a-b)]^{n/2}} \int_0^\infty \frac{t^{(n/2)-1}}{(1+t)^{n/2+n/2}} dt = \frac{2^{n-1}}{(a^2-b^2)^{n/2}} B\left(\frac{n}{2}, \frac{n}{2}\right).$$

(b) Taking $n = 3/2, a = 5$ and $b = 3$, in part (a), we get

$$\int_0^\pi \frac{\sin^{(3/2)-1} x \, dx}{(5+3\cos x)^{3/2}} = \frac{(2)^{(3/2)-1}}{(25-9)^{3/4}} B\left(\frac{3}{4}, \frac{3}{4}\right)$$

$$= \frac{\sqrt{2}}{2^3} \cdot \frac{\Gamma(3/4)\Gamma(3/4)}{\Gamma(3/4+3/4)} = \frac{\sqrt{2}\{\Gamma(3/4)\}^2}{8\Gamma(3/2)}$$

$$= \frac{\sqrt{2}\{\Gamma(3/4)\}^2}{8.1/2\sqrt{\pi}} = \frac{\{\Gamma(3/4)\}^2}{2\sqrt{2\pi}}.$$

❑

UNIT VII

SECTION-A VERY SHORT ANSWER TYPE QUESTIONS

Q.1. Define Reclification.

Ans. The process of finding the length of an arc of a curve between two given points is called reclification.

Q.2. Define surface of revolution.

Ans. When a plane curve is revolved about a certain fixed line lying in its own plane, a surface is generated. This surface is called a surface of revolution.

Q.3. The volume generated by the revolution of an ellipse having semi axes *a* and *b* about a tangent at vertex.

Ans. Let the equation of an ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The centroid of this ellipse is (0, 0) and the area is πab. There are four vertices $(\pm a, 0)$ and $(0, \pm b)$. Now first we revolve the ellipse about tangent at $(a, 0)$, then the distance of the centroid $(0, 0)$ from this tangent is a.

Thus the generated volume = (area of ellipse) × (perimeter of the circle of radius a)

$$=\pi ab \times 2\pi a = 2\pi^2 a^2 b.$$

Similarly, if we revolve the ellipse about the tangent at $(0, b)$, then the distance of the centroid $(0, 0)$ from this tangent is b. Thus the required volume is

= (area of an ellipse) × (perimeter of the circle of radius b)

$$=\pi ab \times 2\pi b = 2\pi^2 ab^2.$$

Q.4. Evaluate the following double integral $\int_0^a \int_0^b (x^2+y^2)\,dx\,dy$.

Ans. We have $\int_0^a \int_0^b (x^2+y^2)\,dx\,dy = \int_0^a \left[x^2 y + \frac{y^3}{3}\right]_{y=0}^{b} dx,$

(integrating w.r.t. y treating x as constant)

$$= \int_0^a \left[bx^2 + \frac{b^3}{3}\right] dx = \left[b\frac{x^3}{3} + \frac{b^3}{3}x\right]_0^a = \frac{ba^3}{3} + \frac{b^3 a}{3} = \frac{1}{3}ab(a^2+b^2).$$

Q.5. Evaluate $\int_1^2 \int_0^x \frac{1}{x^2+y^2}\,dx\,dy$.

Ans. We have $\int_1^2 \int_0^x \frac{dx\,dy}{x^2+y^2} = \int_1^2 \left[\int_0^x \frac{dy}{x^2+y^2}\right] dx$

$$=\int_1^2\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_{y=0}^{x}dx=\int_1^2\left[\frac{1}{x}(\tan^{-1}1-\tan^{-1}(0)\right]dx$$

$$=\frac{\pi}{4}\int_1^2\frac{dx}{x}=\frac{\pi}{4}[\log x]_1^2=\frac{\pi}{4}[\log 2-\log 1]=\frac{1}{4}\pi\log 2.$$

Q.6. Evaluate $\int_1^2\int_0^{y/2} y\,dy\,dx$.

Ans. We have $\int_1^2\int_0^{y/2} y\,dy\,dx=\int_1^2 y[x]_0^{y/2}\,dy=\int_1^2 y\left(\frac{1}{2}y\right)dy$

$$=\frac{1}{2}\int_1^2 y^2\,dy=\frac{1}{2}\left[\frac{1}{3}y^3\right]_1^2=\frac{1}{6}(2^3-1^3)=7/6$$

Q.7. Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,d\theta\,dr$.

Ans. Let $I=\int_0^{\pi}\int_0^{a\sin\theta} r\,d\theta\,dr=\int_0^{a}\left[\int_0^{a\sin\theta} r\,dr\right]d\theta=\frac{a^2}{2}\int_0^{\pi}\sin^2\theta\,d\theta$

$$=\frac{a^2}{2}\times 2\int_0^{\pi/2}\sin^2\theta\,d\theta=a^2\,\frac{\pi}{4}.$$

Q.8. Prove that when x and y are positive and $x+y<h$,
$\iint f'(x+y)x^{l-1}\,y^{-l}\,dx\,dy=\frac{\pi}{\sin l\pi}[f(h)-f(0)]$.

Ans. The given integral

$$I=\iint f'(x+y)\,x^{l-1}\,y^{(1-l)-1}\,dx\,dy,\text{ where } 0<x+y<h$$

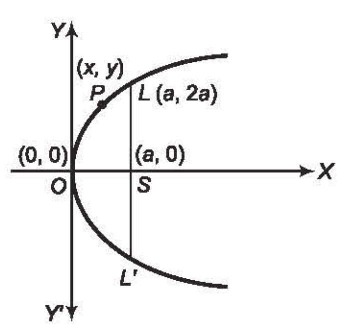
$$=\frac{\Gamma(l)\Gamma(1-l)}{\Gamma(l+1-l)}\int_0^h f'(u)\,u^{l+(1-l)-1}\,du.$$

by Liouville's extension of Dirichlet's theorem

$$=\frac{\Gamma(l)\Gamma(1-l)}{\Gamma(1)}\int_0^h f'(u)\,du=\frac{\pi}{\sin\pi l}[f(u)]_0^h=\frac{\pi}{\sin\pi l}[f(h)-f(0)].$$

SECTION-B SHORT ANSWER TYPE QUESTIONS

Q.1. Find the length of the arc of the parabola $y^2=4ax$ extending from the vertex to an extremity of the latus rectum.



Ans. The given equation of parabola is

$$y^2=4ax. \quad ...(1)$$

The point $O(0,0)$ is the vertex of the parabola and the point $L(a,2a)$ is an extremity of the latus rectum LSL'. We have to find the length of arc OL. Differentiating (1) w.r.t. x, we get $2y(dy/dx)=4a$.

$\therefore$ $$dy/dx = 2a/y \quad \text{or} \quad dx/dy = y/2a.$$

Now $$\left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{4a^2}$$

$$= \frac{1}{4a^2}(4a^2 + y^2). \qquad \text{...(2)}$$

If 's' denotes the arc length of the parabola measured from the vertex O to any point $P(x, y)$ towards the point L, then s increases as y increases. Therefore ds/dy will be positive. So extracting the square root of (2) and keeping the positive sign, we have

$$\frac{ds}{dy} = \frac{1}{2a}\sqrt{(4a^2 + y^2)} \quad \text{or} \quad ds = \frac{1}{2a}\sqrt{(4a^2 + y^2)}\,dy.$$

Let s_1 denote the arc length OL. Then

$$\int_0^{s_1} ds = \int_0^{2a} \frac{1}{2a}\sqrt{(4a^2 + y^2)}\,dy$$

or $$s_1 = \frac{1}{2a}\left[\frac{y}{2}\sqrt{(4a^2 + y^2)} + \frac{4a^2}{2}\log\{y + \sqrt{(4a^2 + y^2)}\}\right]_0^{2a}$$

$$= \frac{1}{2a}[a\sqrt{(4a^2 + 4a^2)} + 2a^2 \log\{2a + \sqrt{(8a^2)}\} - 0 - 2a^2 \log(2a)]$$

$$= \frac{1}{2a}[2\sqrt{2}a^2 + 2a^2 \log\{(2a + 2\sqrt{2}a)/2a\}]$$

$$= \frac{2a^2}{2a}[\sqrt{2} + \log(1 + \sqrt{2})] = a[\sqrt{2} + \log(1 + \sqrt{2})].$$

Q.2. Find the length of the curve $y = \log \dfrac{e^x - 1}{e^x + 1}$ from $x = 1$ to $x = 2$.

Ans. We have $$y = \log\frac{e^x - 1}{e^x + 1} = \log(e^x - 1) - \log(e^x + 1).$$

Differentiating w.r.t. x, we get $\dfrac{dy}{dx} = \dfrac{e^x}{e^x - 1} - \dfrac{e^x}{e^x + 1} = \dfrac{2e^x}{e^{2x} - 1}$.

If s is the arc length of the curve in the direction of x increasing, then

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{2e^x}{e^{2x} - 1}\right)^2 = 1 + \frac{4e^{2x}}{(e^{2x} - 1)^2}$$

$$= \frac{(e^{2x} - 1)^2 + 4e^{2x}}{(e^{2x} - 1)^2} = \left(\frac{e^{2x} + 1}{e^{2x} - 1}\right)^2.$$

$\therefore$ $$\frac{ds}{dx} = \frac{e^{2x} + 1}{e^{2x} - 1}.$$

Integrating w.r.t. from $x=1$ to $x=2$, we get

$$s=\int_1^2 \frac{e^{2x}+1}{e^{2x}-1}\,dx=\int_1^2 \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$$

$$=[\log(e^x-e^{-x})]_1^2=\log(e^2-e^{-2})-\log(e^1-e^{-1})$$

$$=\log\frac{e^2-e^{-2}}{e^1-e^{-1}}=\log\frac{(e^1-e^{-1})(e^1+e^{-1})}{(e^1-e^{-1})} \quad [\because a^2-b^2=(a-b)(a+b)]$$

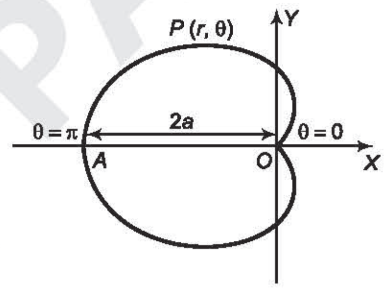
$\therefore$ $$s=\log(e^1+e^{-1}).$$

Q.3. Find the perimeter of the cardioid $r=a(1-\cos\theta)$.

Ans. The given curve is $r=a\,(1-\cos\theta)$. ...(1)

It is symmetrical about the initial line.

We have $r=0$ when $\cos\theta=1$ *i.e.*, $\theta=0$. Also r is maximum when $\cos\theta=-1$ *i.e.*, $\theta=\pi$ and then $r=2a$. As θ increases from 0 to π, r increases from 0 to $2a$. So the curve is as shown in the figure.



By symmetry, the perimeter of the cardioid
$=2\times$ the arc length of the upper half of the cardioid.

Now differentiating (1) w.r.t. θ, we have

$$dr/d\theta=a\sin\theta.$$

We have $$\left(\frac{ds}{d\theta}\right)^2=r^2+\left(\frac{ds}{d\theta}\right)^2=a^2(1-\cos\theta)^2+a^2\sin^2\theta$$

$$=a^2\left(2\sin^2\frac{1}{2}\theta\right)^2+a^2\left(2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta\right)^2$$

$$=4a^2\sin^2\frac{1}{2}\theta\left(\sin^2\frac{1}{2}\theta+\cos^2\frac{1}{2}\theta\right)$$

$$=4a^2\sin^2\frac{1}{2}\theta. \quad ...(2)$$

If s denotes the arc length of the cardioid measured from the cusp O (*i.e.*, the point $\theta=0$) to any point $P(r,\theta)$ in the direction of θ increasing, then s increases as θ increases. Therefore $ds/d\theta$ will be positive. Hence from (2), we have

$$ds/d\theta=2a\sin\frac{1}{2}\theta, \quad \text{or} \quad ds=2a\sin\frac{1}{2}\theta\,d\theta. \quad ...(3)$$

As the cusp O, $\theta=0$ and at the vertex A, $\theta=\pi$.

$\therefore$ the length of the arc $OPA=\int_0^{\pi}2a\sin\frac{1}{2}\theta\,d\theta$

$$=4a\left[-\cos\frac{\theta}{2}\right]_2^{\pi}=-4a\left[\cos\frac{\theta}{2}\right]_0^{\pi}=-4a\,(0-1)=4a.$$

$\therefore$ the perimeter of the cardioid $=2\times4a=8a$.

Q.4. Find the intrinsic equation of the curve $r = a(1+\cos\theta)$. Also prove that $s^2 + 9\rho^2 = 16a^2$.

Ans. We have $\quad r = a\,(1+\cos\theta)$ ...(1)

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2\,(1+\cos\theta)^2 + a^2\sin^2\theta} = 2a\cos\frac{\theta}{2}$$

If s be the length of the arc of the curve (1) measured from pole $(0, 0)$ to any point $P(r, \theta)$ in the direction of θ increasing. Then

$$s = \int_0^{\theta}\left(\frac{ds}{d\theta}\right)d\theta = 2a\int_0^{\theta}\cos\frac{\theta}{2}\,d\theta = 4a\sin\frac{\theta}{2}$$

$$\therefore \quad s = 4a\sin\frac{\theta}{2} \quad ...(2)$$

Also we know that

$$\tan\phi = \frac{r\,d\theta}{dr} = -\frac{a\,(1+\cos\theta)}{a\sin\theta} = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\Rightarrow \quad \phi = \frac{\pi}{2}+\frac{\theta}{2}$$

But, we have $\quad \psi = \theta + \phi = \theta + \frac{\pi}{2} = \frac{\pi}{2} + \frac{3\theta}{2}$

$$\Rightarrow \quad \frac{3\theta}{2} = \left(\psi - \frac{\pi}{2}\right) \quad \Rightarrow \quad \frac{\theta}{2} = \left(\frac{\psi}{3} - \frac{\pi}{6}\right).$$

Putting the value of θ in (2), we get

$$s = 4a\sin\left(\frac{\psi}{3}-\frac{\pi}{6}\right). \quad ...(3)$$

This is required intrinsic equation.

Now, differentiating (3) w.r.t. ψ, we get

$$\rho = \frac{ds}{d\psi} = \frac{4a}{3}\cos\left(\frac{\psi}{3}-\frac{\pi}{6}\right)$$

or

$$3\rho = 4a\cos\left(\frac{\psi}{3}-\frac{\pi}{6}\right). \quad ...(4)$$

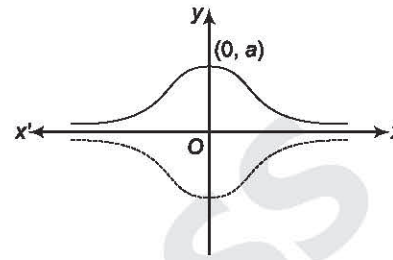
Squaring (3) and (4) and adding, we get

$$s^2 + 9\rho^2 = 16a^2.$$

Q.5. Find the volume of the solid generated by the revolution of the curve $y = a^3/(a^2+x^2)$ about its asymptote.

Ans. Clearly the curve cuts only y-axis at the point $(0, a)$ and $y = 0$, *i.e.*, x-axis is its asymptote. Therefore the solid is generated by the revolution of the curve about x-axis. The tracing of the curve is shown in fig.

Let V be the volume of the solid generated by the revolution of the curve about x-axis from $x=-\infty$ to $x=\infty$, then



$$V=\int_{-\infty}^{\infty} \pi y^2\, dx=\pi \int_{-\infty}^{\infty} \frac{a^6}{(a^2+x^2)^2}\, dx$$

$$[\because\ y=a^3/(a^2+x^2)]$$

$$=2\pi \int_0^{\infty} \frac{a^6}{(a^2+x^2)^2}\, dx.$$

Let us put $x=a\tan\theta$, then $dx=a\sec^2\theta\, d\theta$ and θ varies from $\theta=0$ to $\theta=\pi/2$.

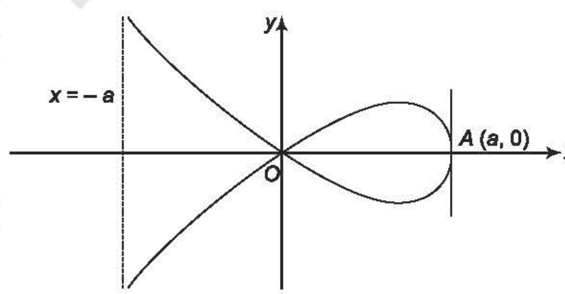
$$\therefore \quad V=2\pi a^6 \int_0^{\pi/2} \frac{a\sec^2\theta\, d\theta}{a^4\sec^4\theta}$$

$$=2\pi a^3 \int_0^{\pi/2} \cos^2\theta\, d\theta=2\pi a^3\left[\frac{(2-1)}{2}\cdot\frac{\pi}{2}\right] \qquad \text{(By Walli's formula)}$$

$$V=\frac{1}{2}\pi^2 a^3.$$

Q.6. Find the volume of a solid formed by the revolution of the loop of the curve $y^2(a+x)=x^2(a-x)$ about x-axis.

Ans. Clearly, the given curve is symmetrical about x-axis and the curve cuts the x-axis only at the points (0, 0) and $(a, 0)$ so the loop exists between these points. The tracing of this curve is shown in fig.



Therefore, the required volume is the volume of a solid formed by the revolution of upper half of the loop of the curve about x-axis where x varies from 0 to a. Then

$$V=\int_0^a \pi y^2\, dx=\pi\int_0^a \frac{x^2(a-x)}{a+x}\, dx \qquad [\because\ y^2(a+x)=x^2(a-x)]$$

$$=\pi\int_0^a \frac{x^2(2a-a-x)}{a+x}\, dx=\pi\int_0^a \frac{2ax^2}{a+x}\, dx-\pi\int_0^a x^2\, dx$$

$$=2a\pi\int_0^a \frac{(x^2-a^2+a^2)}{a+x}\, dx-\pi\int_0^a x^2\, dx$$

$$=2a\pi\left[\int_0^a (x-a)\, dx+a^2\int_0^a \frac{dx}{a+x}\right]-\pi\int_0^a x^2\, dx$$

$$=2a\pi\left[\frac{x^2}{2}-ax+a^2\log(a+x)\right]_0^a-\pi\left[\frac{x^3}{3}\right]_0^a$$

$$=2a\pi\left[\frac{a^2}{2}-a^2+a^2\log 2a-a^2\log a\right]-\frac{\pi a^3}{3}$$

$$=2a\pi\left[-\frac{a^2}{2}+a^2\log\frac{2a}{a}\right]-\frac{\pi a^3}{3}$$

$$=-\pi a^3+2a^3\pi\log 2-\frac{\pi a^3}{3}=2a^3\pi\log 2-\frac{4\pi}{3}a^3$$

$$V=2\pi a^3\left[\log 2-\frac{2}{3}\right].$$

Q.7. Find the volume of the solid generated by the revolution of the tractrix $x=a\cos t+\frac{1}{2}a\log\tan^2(t/2)$, $y=a\sin t$ about its asymptote.

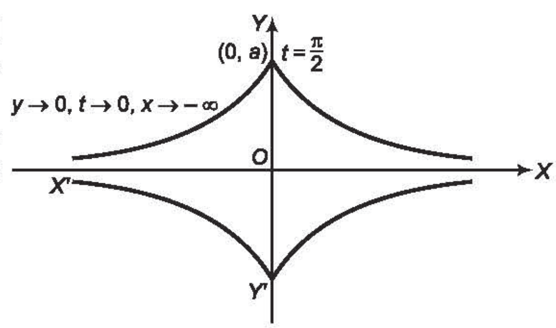
Ans. The given curve is

$$x=a\cos t+\frac{1}{2}a\log\tan^2(t/2),\ y=a\sin t. \qquad \text{...(1)}$$

$$\therefore \quad \frac{dx}{dt}=-a\sin t+\frac{1}{2}a\cdot\frac{1}{\tan^2(t/2)}\cdot 2\tan(t/2)\sec^2(t/2)\cdot\frac{1}{2}$$

$$=-a\sin t+\frac{a}{2\sin(t/2)\cos(t/2)}=-a\sin t+\frac{a}{\sin t}$$

$$=a\frac{(1-\sin^2 t)}{\sin t}=a\frac{\cos^2 t}{\sin t} \qquad \text{...(2)}$$



Now the given curve is symmetrical about both the axes and the asymptote is the line $y=0$ *i.e.*, x-axis.

For the portion of the curve lying in the second quadrant y varies from a to 0, t varies from $\pi/2$ to 0 and x varies from 0 to $-\infty$.

$\therefore$ the required volume

$$=2\int_{-\infty}^{0}\pi y^2\,dx=2\int_0^{\pi/2}\pi y^2\frac{dx}{dt}\cdot dt$$

$$=2\pi\int_0^{\pi/2}a^2\sin^2 t\cdot\frac{a\cos^2 t}{\sin t}\,dt \qquad \text{[From (1) and (2)]}$$

$$=2\pi a^3\int_0^{\pi/2}\cos^2 t\sin t\,dt=2\pi a^3\,\frac{1}{3.1}=\frac{2}{3}\pi a^3.$$

Q.8. Find the surface generated by the revolution of an arc of the catenary $y=c\cosh(x/c)$ about the axis of x.

Ans. The given curve is, $y=c\cosh(x/c)$ ...(1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = c \sinh \frac{x}{h} . \frac{1}{c} = \sinh \frac{x}{c}.$$

$$\therefore \qquad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \sinh^2 \frac{x}{c}\right\}} = \cosh \frac{x}{c}. \qquad ...(2)$$

It the arc be measured from the vertex $(x = 0)$ to any point (x, y), then the required surface formed by the revolution of this arc about x-axis

$$= \int_{x=a}^{0} 2\pi\, y \frac{ds}{dx} dx = 2\pi \int_0^x c \cosh \frac{x}{c} . \cosh \frac{x}{c} dx, \qquad \text{from (1) and (2)}$$

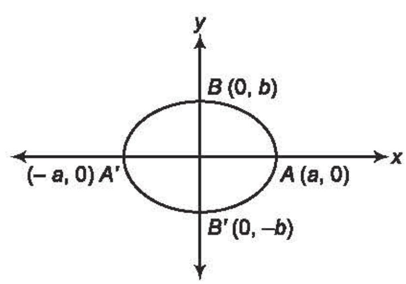
$$= \pi\, c \int_0^x 2 \cosh^2 \frac{x}{c} dx = \pi\, c \int_c^x \left[1 + \cosh \frac{2x}{c}\right] dx$$

$$= \pi\, c \left[x + \frac{c}{2} \sinh \frac{2x}{c}\right]_0^x = \pi\, c \left[x + \frac{c}{2} \sinh \frac{2x}{c}\right]$$

$$= \pi\, c \left[x + c \sinh \frac{x}{c} \cosh \frac{x}{c}\right].$$

Q.9. Prove that the ratio of volumes of the solids generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its major and minor axes is $b : a$.

Ans. About major axis : Let V_1 be the volume of the solid of revolution about major axis AA'. Then



$$V_1 = \int_{-a}^{a} \pi y^2\, dx = \int_{-a}^{a} \pi b^2 \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= \frac{2\pi b^2}{a^2} \int_0^a (a^2 - x^2)\, dx$$

$$= \frac{2\pi b^2}{a^2} \left[a^2 x - \frac{x^3}{3}\right]_0^a = \frac{2\pi b^2}{a^2} \left[\frac{2a^3}{3}\right] = \frac{4\pi}{3} ab^2$$

About minor axis : Let V_2 be the volume of the solid of revolution about minor axis BB', then

$$V_1 = \int_{-b}^{b} \pi x^2\, dy = \pi \int_{-b}^{b} a^2 \left(1 - \frac{y^2}{b^2}\right) dy = 2\pi a^2 \int_0^b \left(1 - \frac{y^2}{b^2}\right) dy$$

$$= \frac{2\pi a^2}{b^2} \int_0^b (b^2 - y^2)\, dy = \frac{2\pi a^2}{b^2} \left[b^2 y - \frac{y^3}{3}\right]_0^b = \frac{2\pi a^2}{b^2} \left[\frac{2b^3}{3}\right] = \frac{4\pi a^2 b}{3}$$

Hence $V_1 : V_2 = \frac{4\pi ab^2}{3} : \frac{4\pi a^2 b}{3} = b : a$

Q.10. Prove that the surface of the solid generated by the revolution of the tractrix $x = a\cos t + \frac{1}{2} a \log \tan^2 \frac{1}{2} t$, $y = a \sin t$ about its asymptote is equal to the surface of a sphere of radius a.

Ans. The given tractric is

$$x = a\cos t + \frac{1}{2} a \log \tan^2 \frac{1}{2} t,\ y = a\sin t.$$

$$\therefore \qquad \frac{dx}{dt} = -a\sin t + a\,\frac{\sec^2 \frac{1}{2}t}{\tan \frac{1}{2}t}\cdot\frac{1}{2} = a\left(-\sin t + \frac{1}{2\sin\frac{1}{2}t\cos\frac{1}{2}t}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = a\,\frac{(-\sin^2 t + 1)}{\sin t} = \frac{a\cos^2 t}{\sin t} \text{ and } \frac{dy}{dt} = a\cos t.$$

Hence
$$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} = \sqrt{\left\{\frac{a^2\cos^4 t}{\sin^2 t} + a^2\cos^2 t\right\}} = \frac{a\cos t}{\sin t}.$$

The given curve is symmetrical about both the axes and the asymptote is the line $y = 0$ *i.e.*, x-axis. For the arc of the curve lying in the second quadrant t varies from 0 to $\frac{1}{2}\pi$.

$$\therefore \text{ the required surface} = 2.\int_0^{\pi/2} 2\pi y \frac{ds}{dt}\,dt$$

$$= 4\pi\int_0^{\pi/2} a\sin t\,.\frac{a\cos t}{\sin t}\,dt = 4\pi a^2\int_0^{\pi/2}\cos t\,dt$$

$$= 4\pi a^2 [\sin t]_0^{\pi/2} = 4\pi a^2$$

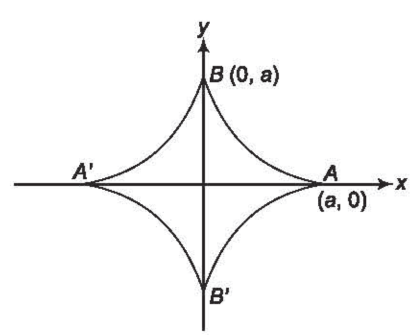
= the surface of a sphere of radius a.

Q.11. Find the surface of the solid generated by the revolution of the curve $x = \cos^3 t$ and $y = a\sin^3 t$ about the x-axis.

Ans. We have $x = a\cos^3 t$ and $y = a\sin^3 t$.

This curve is symmetrical about both the axes.

At $A(a, 0) \Rightarrow t = 0$ and at $B(0, a) \Rightarrow t = \pi/2$.



Now
$$\frac{dx}{dt} = -3a\cos^2 t\sin t$$

$$\frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$\therefore \qquad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= 3a\sqrt{\cos^4 t\sin^2 t + \sin^4 t\cos^2 t} = 3a\sin t\cos t = \frac{3a}{2}\sin 2t.$$

Let S be the surface of a solid of revolution of the given curve about x-axis. Then

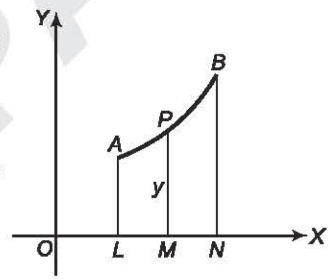
$$S = 2\int_0^{\pi/2} 2\pi y \frac{ds}{dt}\, dt = 4\pi \int_0^{\pi/2} a \sin^3 t \,.\, \frac{3a}{2} \sin 2t\, dt$$

$$= 12\pi a^2 \int_0^{\pi/2} \sin^4 t \cos t\, dt = 12\pi a^2 \left[\frac{\sin^5 t}{5}\right]_0^{\pi/2} = \frac{12\pi a^2}{5}$$

Q.12. State and prove the theorem of pappus and Guldin for surface of a solid of revolution.

Ans. Theorem of Pappus and Guldin for surface of a solid of revolution : If an arc of a plane curve revolves about a straight line in its plane, which does not intersect it, the surface of the solid thus obtained is equal to the arc multiplied by the length of the path described by the centroid of the arc.

Proof : Let l be the length of the arc AB and let it revolve about OX.



Let the abscissae of the extremities A and B of the arc be a and b.

Then the surface generated by the revolution of the arc AB about x-axis is

$$= \int_{x=a}^{x=b} 2\pi\, y\, ds \qquad ...(1)$$

Also we know that (see the chapter on centre of gravity) the ordinate $\bar{y}$, of the centroid of the arc from $x = a$ to $x = b$, of length l, is given by

$$\bar{y} = \frac{\int_{x=a}^{b} y\, ds}{l} \qquad ...(2)$$

From (1) and (2), we get the required surface $= 2\pi\, \bar{y}\, l = l \times 2\pi\, \bar{y}$

= (length of the arc) × (length of the path described by the centroid of the arc).

Q.13. Find the area by double integration the region bounded by circle $x^2 + y^2 = a^2$.

Ans. The area of a small element at any point (x, y) is $dx\, dy$. Now to find the area bounded by the circle $x^2 + y^2 = a^2$, the region of integration R can be expressed as

$$-a \le y \le a, -\sqrt{a^2 - y^2} \le x \le \sqrt{(a^2 - y^2)}.$$

Now, first integration is to be performed w.r. to x regarding y as constant.

∴ the required area

$$= \iint_R dx\, dy = \int_{y=-a}^{a} \int_{x=-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}} 1 \,.\, dy\, dx$$

$$= \int_{-a}^{a} \left[2\int_0^{\sqrt{(a^2-y^2)}} 1 \,.\, dx\right] dy, \text{ by the property of definite integral}$$

$$= 2\int_{-a}^{a} [x]_0^{\sqrt{(a^2-y^2)}}\, dy = 2\int_{-a}^{a} \sqrt{(a^2 - y^2)}\, dy = 2 \,.\, 2\int_0^{a} \sqrt{(a^2 - y^2)}\, dy$$

$$=4\left[\frac{y\sqrt{(a^2-y^2)}}{2}+\frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_0^a$$

$$=4\left[0+\frac{a^2}{2}\sin^{-1}1\right]=4\cdot\frac{1}{2}a^2\cdot\frac{1}{2}\pi=\pi a^2.$$

Q.14. Evaluate :

(i) $\int_0^3\int_1^2 xy(1+x+y)\,dx\,dy$ **(ii)** $\int_0^1\int_0^{\sqrt{(1+x^2)}}\frac{dx\,dy}{1+x^2+y^2}$.

Ans. (i) We have $\int_0^3\int_1^2 xy(1+x+y)\,dx\,dy$

$$=\int_0^3\left[x\cdot\frac{y^2}{2}+x^2\cdot\frac{y^2}{2}+x\cdot\frac{y^3}{3}\right]_{y=1}^2 dx,\ \text{(integrating w.r.t. } y \text{ treating } x \text{ as constant)}$$

$$=\int_0^3\left[\frac{x}{2}(4-1)+\frac{x^2}{2}(4-1)+\frac{x}{3}(8-1)\right]dx=\int_0^3\left[\left(\frac{3}{2}+\frac{7}{3}\right)x+\frac{3}{2}x^2\right]dx$$

$$=\left[\frac{23}{6}\cdot\frac{x^2}{2}+\frac{3}{2}\cdot\frac{x^3}{3}\right]_0^3=\frac{23}{6}\cdot\frac{9}{2}+\frac{27}{2}=\frac{123}{4}=30\frac{3}{4}.$$

(ii) We have $\int_0^1\int_0^{\sqrt{(1+x^2)}}\frac{dx\,dy}{1+x^2+y^2}$

$$=\int_0^1\frac{1}{\sqrt{(1+x^2)}}\left[\tan^{-1}\frac{y}{\sqrt{(1+x^2)}}\right]_{y=0}^{\sqrt{(1+x^2)}}dx,$$

(integrating w.r.t. y treating x as constant)

$$=\int_0^1\frac{1}{\sqrt{(1+x^2)}}[\tan^{-1}1-\tan^{-1}0]\,dx=\frac{\pi}{4}\int_0^1\frac{dx}{\sqrt{(1+x^2)}}$$

$$=\frac{\pi}{4}[\log\{x+\sqrt{(1+x^2)}\}]_0^1=\frac{\pi}{4}\log(1+\sqrt{2}).$$

Q.15. Evaluate $\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,d\theta\,dr$.

Ans. We have $\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,d\theta\,dr=\int_0^{\pi/2}\left[\frac{1}{2}r^2\right]_{a(1-\cos\theta)}^{a\sin\theta}d\theta$

$$=\frac{1}{2}a^2\int_0^{\pi/2}[\sin^2\theta-(1-\cos\theta)^2]\,d\theta$$

$$=\frac{1}{2}a^2\int_0^{\pi/2}[2\cos\theta-\cos^2\theta-(1-\sin^2\theta)]\,d\theta$$

$$=a^2\int_0^{\pi/2}(\cos\theta-\cos^2\theta)\,d\theta=a^2\int_0^{\pi/2}\left[\cos\theta-\frac{1}{2}(1+\cos2\theta)\right]d\theta$$

$$=a^2\left[\sin\theta-\frac{1}{2}\theta-\frac{1}{4}\sin2\theta\right]_0^{\pi/2}=a^2\left[1-\frac{\pi}{4}\right]=\frac{1}{4}a^2\,(4-\pi).$$

Q.16. Evaluate $\iiint z^2\,dx\,dy\,dz$ over the sphere $x^2+y^2+z^2=1$.

Ans. Here the region of integration can be expressed as

$-1\le x\le1, -\sqrt{(1-x^2)}\le y\le\sqrt{(1-x^2)}, -\sqrt{(1-x^2-y^2)}\le z\le\sqrt{(1-x^2-y^2)}$.

$\therefore$ the required triple integral

$$=\int_{-1}^{1}\int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}}\int_{-\sqrt{(1-x^2-y^2)}}^{\sqrt{(1-x^2-y^2)}} z^2\,dx\,dy\,dz$$

$$=\int_{-1}^{1}\int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}}\left[\frac{z^3}{3}\right]_{-\sqrt{(1-x^2-y^2)}}^{\sqrt{(1-x^2-y^2)}}dx\,dy$$

$$=\frac{1}{3}\int_{-1}^{1}\left[\int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}}2\,(1-x^2-y^2)^{3/2}\,dy\right]dx$$

$$=\frac{2}{3}\int_{-1}^{1}\left[\int_{-\pi/2}^{\pi/2}[(1-x^2)\cos^2\theta]^{3/2}.\sqrt{(1-x^2)}.\cos\theta\,d\theta\right]dx$$

[putting $y=\sqrt{(1-x^2)}\sin\theta$ so that $dy=\sqrt{(1-x^2)}\cos\theta\,d\theta$;
also when $y=0, \theta=0$ and when $y=\sqrt{(1-x^2)}, \theta=\pi/2$]

$$=\frac{2}{3}\int_{-1}^{1}\left[2.\int_0^{\pi/2}(1-x^2)^2\cos^4\theta\,d\theta\right]dx$$

$$=\frac{4}{3}\int_{-1}^{1}(1-x^2)^3.\frac{3.1}{4.2}.\frac{\pi}{2}\,dx=\frac{\pi}{4}\int_{-1}^{1}(1-x^2)^2\,dx$$

$$=\frac{\pi}{4}.2\int_0^1(1-2x^2+x^4)\,dx=\frac{\pi}{2}\left[x-\frac{2}{3}x^3+\frac{1}{5}x^5\right]_0^1$$

$$=\frac{\pi}{2}\left[1-\frac{2}{3}+\frac{1}{5}\right]=\frac{\pi}{2}.\frac{8}{15}=\frac{4\pi}{15}$$

Q.17. Evaluate $\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$.

Ans. The given integral

$$I=\int_{x=0}^{1}\int_0^{\sqrt{1-x^2}} xy\left(\frac{1}{2}z^2\right)_0^{\sqrt{1-x^2-y^2}}dx\,dy$$

$$=\frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} xy\,(1-x^2-y^2)\,dx\,dy$$

$$=\frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} x\,[y\,(1-x^2)-y^3]\,dx\,dy$$

$$=\frac{1}{2}\int_{x=0}^{1} x\left[\frac{1}{2}(1-x^2)\,y^2-\frac{1}{4}y^4\right]_0^{\sqrt{1-x^2}} dx$$

$$=\frac{1}{2}\int_0^1 x\left[\frac{1}{2}(1-x^2)(1-x^2)-\frac{1}{4}(1-x^2)^2\right]dx$$

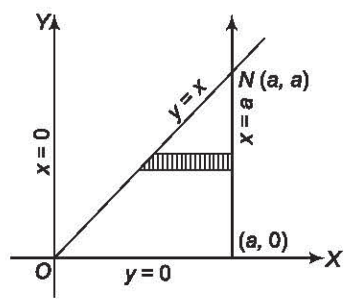
$$=\frac{1}{2}\int_0^1 x\left(\frac{1}{2}-\frac{1}{4}\right)(1-x^2)^2\,dx$$

$$=\frac{1}{8}\int_0^1 (x-2x^3+x^5)\,dx=\frac{1}{8}\left[\frac{1}{2}x^2-\frac{1}{2}x^4+\frac{1}{6}x^6\right]_0^1$$

$$=\frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right)=\frac{1}{48}.$$

Q.18. Change the order of integration in the double integral $\int_0^a\int_0^x f(x,y)\,dx\,dy$.

Ans. In the given integral the limits of integration are given by the straight lines $y=0$, $y=x$, $x=0$ and $x=a$. Draw these lines bounding the region of integration in the same figure. We observe that the region of integration is the area *ONM*. In the given integral, the limits of integration of y being variable. we are required to integrate first w.r.t. y regarding x as constant and then w.r.t. x.



To reverse the order of integration, we have to integrate first w.r.t. x regarding y as constant and then w.r.t. y. This is done by dividing the area *ONM* into strips parallel to the x-axis. Let us take strips parallel to the x-axis starting from the line *ON* (*i.e.*, $y=x$) and terminating on the line *MN* (*i.e.*, $x=a$). Thus for this region *ONM*, x varies from y to a and y varies from 0 to a. Hence by changing the order of integration, we have

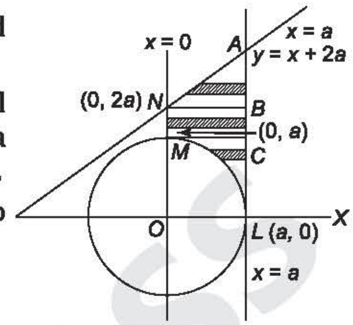
$$\int_0^a\int_0^x f(x,y)\,dx\,dy=\int_0^a\int_y^a f(x,y)\,dy\,dx.$$

Q.19. Change the order of integration in $\int_0^a\int_{\sqrt{a^2-x^2}}^{x+2a} f(x,y)\,dx\,dy$.

Ans. The area of integration is bounded by the curves $y=\sqrt{a^2-x^2}$

i.e., $$x^2+y^2=a^2.$$

This is the equation of the circle with centre (0, 0) and radius a. Also $y=x+2a$ represent a straight line which passing through $(0, 2a)$, *i.e.*, the Y-axis and the line $x=a$ which is parallel to Y-axis.



We draw the curves $x^2 + y^2 = a^2$, $y = x + 2a$, $x = 0$ and $x = a$. We observe that the region of integration is area *MLANM.*

To change the order of integration we draw a strip parallel to *x*-axis. Draw the lines *MC* and *MB* parallel to *X*-axis. So the area of integration is divided into three portions *MLEC, NNCB* and *NAB.*

For region *MLC, x* varies from $x^2 + y^2 = a^2$ circle's arc to line $x = a$

or $\quad x = \sqrt{a^2 - y^2}$ to a and y varies from 0 to a.

For region *NMCB, x* varies from 0 to a and y varies from a to $2a$.

For region *NBA, x* varies from $y - 2a$ to a and y varies from $2a$ to $3a$.

So, the given integral transform to

$$\int_0^a \int_{\sqrt{a^2-y^2}}^a f(x, y)\, dy\, dx + \int_a^{2a} \int_0^a f(x, y)\, dy\, dx + \int_{2a}^{3a} \int_{y-2a}^a f(x, y)\, dy\, dx.$$

Q.20. Find the volume of the tetrahedron bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the co-ordinate planes.

Ans. The volume of a small element at a point $(x, y, z) = dx\, dy\, dz$.

$\therefore$ the volume of the given tetrahedron $= \iiint dx\, dy\, dz$

where the integral is extended to all positive values of variables x, y, z.

Put $x/a = u, y/b = v, z/c = w$ subject to the condition so that

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \le 1$$

$$dx = a\, du, dy = b\, dv, dz = c\, dw$$

then the required volume $= \iiint abc\, du\, dv\, dw$ where $u + v + w \le 1$

$$= abc \iiint u^{1-1} v^{1-1} w^{1-1}\, du\, dv\, dw$$

$$= abc\, \frac{(\Gamma(1))^3}{\Gamma(1+1+1+1)} \qquad \text{[By Dirichlet's integral]}$$

$$= abc\, \frac{1}{\Gamma(4)} = \frac{abc}{3.2.1} = \frac{abc}{6}.$$

Q.21. Find the value of $\iiint \log(x + y + z)\, dx\, dy\, dz$, the integral extending over all positive values of x, y, z subject to the condition $x + y + z < 1$.

Ans. Here the integral is to be extended for all positive values of *x, y* and *z* such that $0 < x + y + z < 1$.

$\therefore$ the required integral

$$= \iiint \log(x + y + z)\, dx\, dy\, dz, \text{ where } 0 < x + y + z < 1$$

$$= \iiint \log(x + y + z)\, x^{1-1}\, y^{1-1}\, z^{1-1}\, dx\, dy\, dz$$

$$=\frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1)}\int_0^1 (\log u)\, u^{1+1+1-1}\, du,$$

by Liouville's extension of Dirichlet's theorem

$$=\frac{1}{\Gamma(3)}\int_0^1 u^2 \log u\, du, \qquad [\because \Gamma(1)=1]$$

$$=\frac{1}{2.1}\left[\left((\log u).\frac{u^3}{3}\right)_0^1-\int_0^1 \frac{1}{u}.\frac{u^3}{3}\, du\right],$$

integrating by parts taking u^2 as the second function

$$=\frac{1}{2}\left[0-\frac{1}{3}\lim_{u\to 0} u^3 \log u-\frac{1}{3}\int_0^1 u^2\, du\right]$$

$$=-\frac{1}{6}\left[\frac{u^3}{3}\right]_0^1, \qquad \left[\because \lim_{u\to 0} u^3 \log u=0\right]$$

$$=-\frac{1}{18}.$$

Note : $\lim\limits_{u\to 0} u^3 \log u=\lim\limits_{u\to 0}\dfrac{\log u}{1/u^3}=\lim\limits_{u\to 0}\dfrac{1/u}{-3/u^4}=\lim\limits_{u\to 0}-\dfrac{1}{3}u^3=0.$

Q.22. Evaluate $\iiint x^{\alpha}\, y^{\beta}\, z^{\gamma}\,(1-x-y-z)^{\gamma}\, dx\, dy\, dz$ over the interior of tetrahedron formed by the co-ordinate planes and the plane $x+y+z=1$.

Ans. The region of integration is bounded by the plane $x=0, y=0, z=0$ and $x+y+z=1$. So, the variable x, y, z take all positive value subject to the condition $0<x+y+z<1$.

Therefore the given integral

$$=\iiint x^{(\alpha+1)-1}\, y^{(\beta+1)-1}\, z^{(\gamma+1)-1}\,[1-(x+y+z)]^{\lambda}\, dx\, dy\, dz$$

$$=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)}\int_0^1 u^{\alpha+1+\beta+1+\gamma+1-1}\,(1-u)^{\lambda}\, du$$

[By Liouville's extension of Dirichlet's theorem]

$$=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)}\int_0^1 u^{(\alpha+\beta+\gamma+3)-1}\,(1-u)^{(\lambda+1)-1}\, du$$

$$=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)}B(\alpha+\beta+\gamma+3, \lambda+1)$$

$$=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)}.\frac{\Gamma(\alpha+\beta+\gamma+3)\Gamma(\lambda+1)}{\Gamma(\alpha+\beta+\gamma+\lambda+4)}$$

$$=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\lambda+1)}{\Gamma(\alpha+\beta+\gamma+\lambda+4)}$$

SECTION-C LONG ANSWER TYPE QUESTIONS

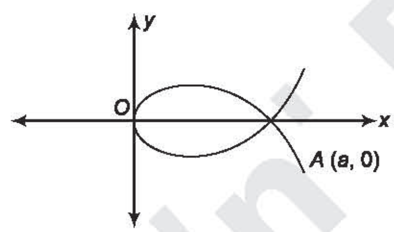
Q.1. Find the length of the loop of the curve $3ay^2 = x(x-a)^2$.

Ans. The equation of the curve is $3ay^2 = x(x-a)^2$. ...(1)

This curve is symmetrical about the x-axis and passes through the origin. The tangent at $(0, 0)$ is the y-axis. The curve cuts the x-axis only at the point $(a, 0)$.

Now differentiating (1) w.r.t. x, we get $\frac{dy}{dx} = \frac{3x-a}{2\sqrt{3ax}}$. ...(2)

$\therefore$ at $(a, 0)$ $\frac{dy}{dx} = \pm\frac{1}{\sqrt{3}}$. Thus at $(a, 0)$ tangents make the angles $\pi/6$ and $-\pi/6$ with positive x-axis. Therefore, we have



If s is the arc length measured from 0 to any point on the curve in the direction of x increasing. Then we will take $\frac{ds}{dx}$ positive.

$$\frac{ds}{dx} = \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]} = \sqrt{\left[1+\frac{(3x-a)^2}{12ax}\right]} = \sqrt{\left(\frac{12ax+9x^2+a^2-6xa}{12ax}\right)}$$

$$\frac{ds}{dx} = \frac{3x+a}{2\sqrt{3ax}}.$$

If s_1 denotes the length of the loop of the curve between the points $x=0$ to $x=a$. Therefore, we have

$$s_1 = 2\int_0^a \left(\frac{ds}{dx}\right).dx$$

or

$$s_1 = 2\int_0^a \frac{3x+a}{2\sqrt{3ax}}\,dx = \frac{1}{\sqrt{3a}}\int_0^a \frac{3x+a}{\sqrt{x}}\,dx$$

$$= \frac{1}{\sqrt{3a}}\int_0^a (3x^{1/2}+ax^{-1/2})\,dx = \frac{1}{\sqrt{3a}}[2x^{3/2}+2ax^{1/2}]_0^a$$

$$= \frac{1}{\sqrt{3a}}[2a\sqrt{a}+2a\sqrt{a}] = \frac{4a\sqrt{a}}{\sqrt{3}\sqrt{a}}$$

$$s_1 = \frac{4a}{\sqrt{3}}.$$

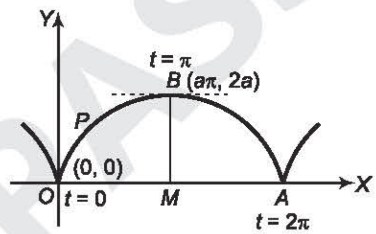
Q.2. Show that $8a$ is the length of an arc of the cycloid whose equations are $x = a(t - \sin t),\ y = a(1 - \cos t)$.

Ans. The given equations of the cycloid are $x = a\,(t - \sin t),\ y = a\,(1 - \cos t)$.

We have $dx/dt = a\,(1 - \cos t)$, and $dy/dt = a \sin t$.

$$\therefore \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a\,(1 - \cos t)} = \frac{2 \sin \frac{1}{2} t \cos \frac{1}{2} t}{2 \sin^2 \frac{1}{2} t} = \cot \frac{1}{2} t.$$

Now $y = 0$ when $\cos t = 1$ *i.e.,* $t = 0$. At $t = 0,\ x = 0,\ y = 0$ and $dy/dx = \infty$. Thus the curve passes through the point (0, 0) and the tangent there is perpendicular to the x-axis.



Again y is maximum when $\cos t = -1$ *i.e.,* $t = \pi$. When $t = \pi,\ x = a\pi,\ y = 2a,\ dy/dx = 0$. Thus at the point $(a\pi, 2a)$ the tangent to the curve is parallel to the x-axis.

Also in this curve y cannot be negative. Thus an arc OBA of the given cycloid is an shown in the figure. It is symmetrical about the line BM which is the axis of the cycloid.

$$\text{We have} \quad \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \{a\,(1 - \cos t)\}^2 + (a \sin t)^2$$

$$= a^2 \left\{\left(2 \sin^2 \frac{1}{2} t\right)^2 + \left(2 \sin \frac{1}{2} t \cos \frac{1}{2} t\right)^2\right\}$$

$$= 4a^2 \sin^2 \frac{1}{2} t \left(\sin^2 \frac{1}{2} t + \cos^2 \frac{1}{2} t\right) = 4a^2 \sin^2 \frac{1}{2} t. \qquad \text{...(1)}$$

If s denotes the arc length of the cycloid measured from the cusp O to any point P towards the vertex B, then s increases as t increases. Therefore ds/dt will be taken with positive sign. So taking square root of both sides of (1), we have

$$ds/dt = 2a \sin \frac{1}{2} t \quad \text{or} \quad ds = 2a \sin \frac{1}{2} t\, dt.$$

As the cusp O, $t = 0$, and at the vertex B, $t = \pi$.

Now the length of the arc $OBA = 2 \times$ length of the arc OB

$$= 2 \int_0^{\pi} 2a \sin \frac{1}{2} t\, dt = 4a \left[-2 \cos \frac{1}{2} t\right]_0^{\pi} = -8a \left[\cos \frac{1}{2} t\right]_0^{\pi}$$

$$= -8a\,[0 - 1] = 8a.$$

Q.3. For the ellipse $x = a \cos \phi,\ y = b \sin \phi$, show that $ds = a\sqrt{1 - e^2 \cos^2 \phi}\, d\phi$. Hence show that the whole length of the ellipse is

$$2a\pi \left\{1 - \left(\frac{1}{2}\right)^2 . \frac{e^2}{1} . \left(\frac{1}{2} . \frac{3}{4}\right) . \frac{e^4}{3} - \left(\frac{1}{2} . \frac{3}{4} . \frac{5}{6}\right) . \frac{e^6}{5} \ldots\ldots\right\}$$

where e is the eccentricity of the ellipse.

Ans. Here, the given equation of ellipse is

$$x = a\cos\phi,\ \ y = b\cos\phi \qquad ...(1)$$

$$\Rightarrow \qquad \frac{dx}{d\phi} = -a\sin\phi \ \text{ and } \ \frac{dy}{d\phi} = b\cos\phi$$

Now,

$$\frac{dx}{d\phi} = \sqrt{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2}$$

$$= \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}$$

$$= \sqrt{a^2\sin^2\phi + a^2(1-e^2)\cos^2\phi}$$

$$[\because b^2 = a^2(1-e^2) \text{ for ellipse (1)}]$$

$$= a\sqrt{\sin^2\phi + \cos^2\phi - e^2\cos^2\phi}$$

$$= a\sqrt{1-e^2\cos^2\phi}$$

$$ds = a\sqrt{1-e^2\cos^2\phi}\,d\phi$$

For ellipse (1) : Ellipse is symmetrical about both the axes and ϕ varies from 0 to $1/2\pi$. Now the whole length of the ellipse

$$= 4\int_{\phi=0}^{\phi=\pi/2} ds = 4a\int_0^{\pi/2}\sqrt{1-e^2\cos^2\phi}\,d\phi$$

$$= 4a\int_0^{\pi/2}(1-e^2\cos^2\phi)^{1/2}\,d\phi$$

$$= 4a\int_0^{\pi/2}\left[1 + \frac{1}{2}(-e^2\cos^2\phi) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(-e^2\cos^2\phi)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-e^2\cos^2\phi)^2 + \ldots\right]d\phi$$

$$= 4a\int_0^{\pi/2}\left[1 - \frac{1}{2}e^2\cos^2\phi - \frac{1}{2.4}e^x\cos^4\phi - \frac{1.3}{2.4.6}e^6\cos^6\phi \ldots.\right]d\phi$$

$$= 4a\left[\int_0^{\pi/2}d\phi - \frac{1}{2}e^2\int_0^{\pi/2}\cos^2\phi\,d\phi - \frac{1}{2.4}e^4\int_0^{\pi/2}\cos^4\phi\,d\phi - - \frac{1.3}{2.4.6}e^4\int_0^{\pi/2}\cos^6\phi\,d\phi - \ldots..\right]$$

$$=4a\left[\frac{\pi}{2}-\frac{1}{2}e^2.\frac{1}{2}.\frac{\pi}{2}-\frac{1}{2.4}e^4.\frac{3}{4}.\frac{1}{2}\frac{\pi}{4}-\frac{1.3}{2.4.6}e^6.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}-....\right]$$

$$=4a\frac{\pi}{2}\left[1-\frac{1}{2}.\frac{1}{2}e^2-\frac{1.3}{2.4.2.6}e^4-\frac{1.3.5.3.1}{2.4.6.2.4.6}e^6-....\right]$$

$$=2a\pi\left[1-\left(\frac{1}{2}\right)^2e^2-\left(\frac{1}{2}.\frac{3}{4}\right)^2\frac{e^4}{3}-\left(\frac{1.3.5}{2.4.6}\right)^2\frac{e^6}{5}-.....\right]$$

Q.4. Prove the formula $s=\int\frac{r\,dr}{\sqrt{(r^2-p^2)}}$.

Show that the arc of the curve $p^2(a^4+r^4)=a^4r^2$ between the limits $r=b, r=c$ is equal in length to the arc of the hyperbola $xy=a^2$ between the limits $x=b, x=c$.

Ans. From differential calculus, we know that $\tan\phi=r\frac{d\theta}{dr}$ and $\frac{ds}{dr}=\sqrt{\left[1+\left(r\frac{d\theta}{dr}\right)^2\right]}$.

$$\therefore \quad \frac{ds}{dr}=\sqrt{(1+\tan^2\phi)}=\sqrt{(\sec^2\phi)}=\sec\phi$$

$$=\frac{1}{\cos\phi}=\frac{1}{\sqrt{(1-\sin^2\phi)}}=\frac{1}{\sqrt{\{1-(p^2/r^2)\}}} \qquad [\because p=r\sin\phi]$$

$$=\frac{r}{\sqrt{(r^2-p^2)}}.$$

Thus $ds=\frac{r}{\sqrt{(r^2-p^2)}}dr.$

Integrating between the given limits, we get $s=\int\frac{r}{\sqrt{(r^2-p^2)}}dr.$ ...(1)

Now the given curve is $p^2(a^4+r^4)=a^4r^2$ or $p^2=a^4r^2/(a^4+r^4)$.

We have $r^2-p^2=r^2-\frac{a^4r^2}{(a^4+r^4)}=\frac{r^6}{(a^4+r^4)}.$ ...(2)

Therefore from (1), the arc of the given curve between the limits $r=b, r=c$ is

$$=\int_b^c\frac{r\,dr}{\sqrt{(r^2-p^2)}}=\int_b^c\frac{r\,dr}{\sqrt{\{(r^6/(a^4+r^4)\}}} \qquad \text{[From (2)]}$$

$$=\int_b^c\frac{r\sqrt{(a^4+r^4)}}{r^3}dr=\int_b^c\frac{\sqrt{(a^4+r^4)}}{r^2}dr. \qquad ...(3)$$

Also, for the hyperbola $xy=a^2$ *i.e.,* $y=a^2/x, dy/dx=-a^2/x^2$.

$\therefore$ the arc length of the hyperbola $xy = a^2$ between the limits $x = b, x = c$

$$= \int_b^c \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx = \int_b^c \sqrt{\left\{1 + \frac{a^4}{x^4}\right\}}\, dx$$

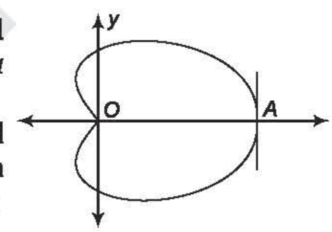
$$= \int_b^c \frac{\sqrt{(x^4 + a^4)}}{x^2}\, dx = \int_b^c \frac{\sqrt{(r^4 + a^4)}}{r^2}\, dr$$ [Changing the variable from x to r by a property of definite integrals]

$$= \int_b^c \frac{\sqrt{(a^4 + r^4)}}{r^2}\, dr. \qquad ...(4)$$

From (3) and (4) we observe that the two length are equal.

Q.5. Find the volume of the solid generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line.

Ans. Obviously, the curve is symmetrical about the initial line and $r = 0$ when $\cos\theta = -1$, *i.e.*, $\theta = \pi$ and the maximum value of $r = 2a$ when $\cos\theta = 1$, *i.e.*, $\theta = 0$. Thus the tracing of the curve is given in fig.



Therefore, the required volume is the volume of a solid generated by the revolution of the upper half of the curve between $\theta = 0$ to $\theta = \pi$ about initial line (x-axis). Let this volume be V. Then

$$V = \int_{\theta=0}^{\theta=\pi} \pi y^2 \frac{dx}{d\theta}.\, d\theta$$

(As θ increases x decreases so $\frac{dx}{d\theta}$ will have to take negative)

$$= \pi \int_0^\pi (r\sin\theta)^2 . \frac{d}{d\theta}(r\cos\theta)\, d\theta \qquad (\because x = r\cos\theta, y = r\sin\theta)$$

$$= -\pi \int_0^\pi a^2 (1 + \cos\theta)^2 \sin^2\theta \frac{d}{d\theta}[a(1 + \cos\theta)\cos\theta]\, d\theta \qquad [\because r = a(1 + \cos\theta)]$$

$$= -\pi a^3 \int_0^\pi (1 + \cos\theta)^2 \sin^2\theta . (-\sin\theta - 2\cos\theta \sin\theta)\, d\theta$$

$$= +\pi a^3 \int_0^\pi (1 + \cos^2\theta + 2\cos\theta)(1 + 2\cos\theta) . \sin^3\theta\, d\theta$$

$$= +\pi a^3 \int_0^\pi (\sin^3\theta + 4\cos\theta \sin^3\theta + 5\cos^2\theta \sin^3\theta + 2\cos^3\theta \sin^3\theta)\, d\theta$$

$$= +\pi a^3 \left[\int_0^\pi \sin^3\theta\, d\theta + 4\int_0^\pi \cos\theta \sin^3\theta\, d\theta + 5\int_0^\pi \cos^2\theta \sin^3\theta\, d\theta + 2\int_0^\pi \cos^3\theta \sin^3\theta\, d\theta\right]$$

$$= \pi a^3 \left[\int_0^\pi \sin^3\theta\, d\theta + 5\int_0^\pi \cos^2\theta \sin^3\theta\, d\theta\right]$$

(The second and fourth integral vanish by the property of definite integral)

$$=\pi a^3\left[2\int_0^{\pi/2}\sin^3\theta\,d\theta+10\int_0^{\pi/2}\cos^2\theta\sin^3\theta\,d\theta\right]$$

(By the property of definite integral)

$$=\pi a^3\left[2.\frac{(3-1)}{3.1}.1+10.\frac{(2-1)(3-1)}{5.3.1}.1\right]$$ (By Walli's formula)

$$=\pi a^3\left[\frac{4}{3}+\frac{4}{3}\right]=\frac{8}{3}\pi a^3.$$

$$\therefore\quad V=\frac{8}{3}\pi a^3.$$

Aliter : The required volume is also taken as

$$V=\int_0^{\pi}\frac{2}{3}\pi r^3\sin\theta\,d\theta=\frac{2}{3}\pi\int_0^{\pi}a^3(1+\cos\theta)^3\sin\theta\,d\theta$$

$$=\frac{2}{3}\pi a^3\int_0^{\pi}(1+\cos\theta)^3\sin\theta\,d\theta$$

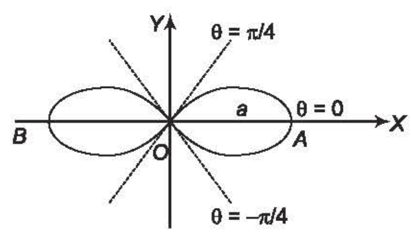
$$=\frac{2}{3}\pi a^3\left[-\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi}=\frac{2}{3}\pi a^3\left[\frac{16}{4}\right].$$

$$\therefore\quad V=\frac{8}{3}\pi a^3.$$

Q.6. Find the surface of the solid generated by the revolution of the lemniscate $r^2=a^2\cos2\theta$ about the initial line.

Ans. The given curve is $r^2=a^2\cos2\theta$. ...(1)

Differentiating (1) w.r.t. θ, we get



$$2r\frac{dr}{d\theta}=-2a^2\sin2\theta\quad\text{or}\quad\frac{dr}{d\theta}=\frac{-a^2\sin2\theta}{r}.$$

$$\therefore\quad\frac{ds}{d\theta}=\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}$$

$$=\sqrt{\left\{a^2\cos2\theta+\frac{a^4\sin^2 2\theta}{r^2}\right\}}=\frac{1}{r}\sqrt{\{r^2.a^2\cos2\theta+a^4\sin^2 2\theta\}}$$

$$=\frac{1}{r}\sqrt{\{a^4\cos^2 2\theta+a^4\sin^2 2\theta\}},\qquad[\because r^2=a^2\cos2\theta]$$

$$=a^2/r.\qquad ...(2)$$

The given curve is symmetrical about the initial line and about the pole.

Putting $r=0$ in (1), we get $\cos2\theta=0$ giving $2\theta=\pm\frac{1}{2}\pi$ *i.e.,* $\theta=\pm\frac{1}{4}\pi$.

Therefore one loop of the curve lies between $\theta = -\frac{1}{4}\pi$ and $\theta = \frac{1}{4}\pi$.

There are two loops in the curve and for the upper half of one of these two loops θ varies from 0 to $\frac{1}{4}\pi$.

$\therefore$ the required surface $= 2 \times$ the surface generated by the revolution of one loop

$$= 2 . \int_0^{\pi/4} 2\pi\, y \frac{ds}{d\theta}\, d\theta, \text{ where } y = r \sin\theta$$

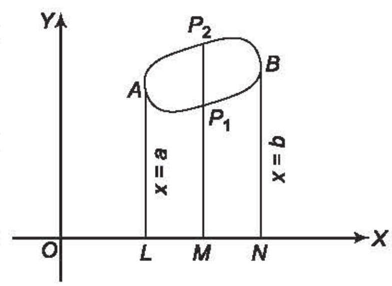
$$= 4\pi \int_0^{\pi/4} r \sin\theta . \frac{a^2}{r}\, d\theta \qquad \text{[From (2)]}$$

$$= 4\pi a^2 \int_0^{\pi/4} \sin\theta\, d\theta = 4\pi a^2 [-\cos\theta]_0^{\pi/4}$$

$$= 4\pi a^2 [-(1/\sqrt{2}) + 1] = 4\pi a^2 [1 - (1/\sqrt{2})].$$

Q.7. State and prove the theorem of Pappus and Guldin for volume of a solid of Revolution.

Ans. Theorem of Pappus and Guldin for volume of a solid of Revolution : If a closed plane curve revolves about a straight line in its place which does not intersect it, the volume of the ring thus obtained is equal to the area of the region enclosed by the curve multiplied by the length of the path described by the centroid of the region.



Proof : Let AP_1BP_2A be the closed plane curve and let it roatet about the axis of x.

Let AL $(x = a)$ and BN $(x = b)$ be the tangents to the curve parallel to the y-axis $(a < b)$. Also let any ordinate meet the curve at P_1, P_2 and let $MP_1 = y_1, MP_2 = y_2$ so that y_1, y_2 are function of x.

Now volume of the ring generated by the revolution of the closed curve AP_1BP_2A about the axis of x

$=$ volume generated by the area $ALNBP_2A$

$-$ volume generated by the area $ALNBP_1A$

$$= \pi \int_a^b y_2^2\, dx - \pi \int_a^b y_1^2\, dx = \pi \int_a^b (y_2^2 - y_1^2)\, dx. \qquad ...(1)$$

Also if $\bar{y}$ be the ordinate of the centroid of the area of the closed curve, then

$$\bar{y} = \frac{\int_a^b \frac{1}{2}(y_1 + y_2)(y_2 - y_1)\, dx}{A} = \frac{\frac{1}{2}\int_a^b (y_2^2 + y_1^2)\, dx}{A} \qquad ...(2)$$

where A is the area of the closed curve. [See the chapter on centre of gravity]

Hence from (1) and (2), the required volume $= 2\pi A \bar{y} = A \times 2\pi \bar{y}$

$=$ area of the closed curve $\times$ circumference of the circle of radius $\bar{y}$

$=$ (area of the curve) $\times$ (length of the arc described by the centroid of the region bounded by the closed curve).

Q.8. State and prove Dirichlet's theorem for three variables.

Ans. Dirichlet's theorem for three variables : If l, m, n are all positive, then the triple integral

$$\iiint x^{l-1}\ y^{m-1} z^{n-1}\ dx\, dy\, dz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)},$$

where the integral is extended to all positive values of the variables x, y and z subject to the condition $x+y+z \le 1$.

Proof : Let us first consider the double integral

$$I_2 = \iint x^{l-1}\ y^{m-1}\ dx\, dy,$$

where the integral is extended to all positive values of the variables x and y subject to the condition $x + y \le 1$.

Obviously the region of integration of I_2, in the 2-dimensional Euclidean space, is bounded by the straight lines $x=0, y=0$ and $x+y=1$. The limits of integration for this region can be expressed as $0 \le x \le 1, 0 \le y \le 1-x$.

$$\therefore \quad I_2 = \int_{x=0}^{1} \int_{y=0}^{1-x} x^{l-1}\ y^{m-1}\ dx\, dy = \int_0^1 x^{l-1} \left[\frac{y^m}{m}\right]_0^{1-x} dx$$

$$= \int_0^1 \frac{1}{m} x^{l-1}\ (1-x)^m\ dx = \frac{1}{m}\int_0^1 x^{l-1}\ (1-x)^{m+1-1}\ dx$$

$$= \frac{1}{m}\mathbf{B}(l, m+1), \quad \text{by the def. of Beta function}$$

$$= \frac{1}{m}\frac{\Gamma(l)\Gamma(m+1)}{\Gamma(l+m+1)} = \frac{1}{m}\frac{\Gamma(l).m\Gamma(m)}{\Gamma(l+m+1)}, \qquad [\because \Gamma(n+1) = n\Gamma(n)]$$

$$= \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)}. \qquad \text{...(1)}$$

The is Dirichelt's theorem for two variables.

Now consider the double integral $U_2 = \iint x^{l-1}\ y^{m-1}\ dx\, dy,$

where the integral is extended to all positive values of the variables x and y subject to the condition $x + y \le h$.

We have $\quad x + y \le h \Rightarrow \dfrac{x}{h} + \dfrac{y}{h} \le 1.$

So putting $x/h = u$ and $y/h = v$ so that $dx = h\, du$ and $dy = h\, dv$, the integral U_2 becomes

$$U_2 = \iint (hu)^{l-1}\ (hv)^{m-1}\ h^2\ du\, dv$$

$$= h^{l+m} \iint u^{l-1}\ v^{m-1}\ du\, dv, \text{ where } u+v \le 1$$

$$= h^{l+m}\frac{\Gamma(L)\Gamma(m)}{\Gamma(l+m+1)}, \text{ by (1).} \qquad \text{...(2)}$$

Now we consider the triple integral

$$I_3 = \iiint x^{l-1}\ y^m\ -1\, z^{n-1}\ dx\, dy\, dz,$$

subject to the condition $x+y+z \le 1$ *i.e.*, $y+z \le 1-x$ and $0 \le x \le 1$.

We have $$I_3 = \int_{x=0}^{1} \left[\iint y^{m-1} z^{n-1} \, dy \, dz \right] x^{l-1} \, dx, \text{ where } y + z \leq 1 - x$$

$$= \int_0^1 (1-x)^{m+n} \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} x^{l-1} \, dx, \quad \text{by using (2)}$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \int_0^1 x^{l-1} (1-x)^{m+n+1-1} \, dx$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \mathbf{B}(l, m+n+1) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \cdot \frac{\Gamma(l)\Gamma(m+n+1)}{\Gamma(l+m+n+1)}$$

$$= \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}, \quad \text{which proves the required result.}$$

Q.9. State and prove Dirichlet's theorem for *n* variables.

Ans. Dirichlet's theorem for *n* variables :

Statement : $$\iint \ldots \int x_1^{l_1-1} x_2^{l_2-1} \ldots x_n^{l_n-1} \, dx_1 \, dx_2 \ldots dx_n = \frac{\Gamma(l_1)\Gamma(l_2)\ldots\Gamma(l_n)}{\Gamma(1+l_1+l_2+\ldots+l_n)}$$

where the integral is extended to all positive values of the variables $x_1, x_2, \ldots x_n$ subject to the condition $x_1 + x_2 + \ldots + x_n \leq 1$.

Proof : We shall prove the theorem by mathematical induction.

First we prove the theorem for 2-variables, *i.e.*, $n = 2$.

Let us consider the integral

$$I_2 = \iint x_1^{l_1-1} x_2^{l_2-1} \, dx_1 \, dx_2.$$

Condition $x_1 + x_2 \leq 1$

Now, using previous theorem, we have

$$I_2 = \frac{\Gamma(l_1)\Gamma(l_2)}{\Gamma(1+l_1+l_2)} \quad \ldots(1)$$

Equation (1) is true for two variables. Now assume that theorem is true for *n* variables. Therefore

$$I_n = \iint \ldots \int x_1^{l_1-1} x_2^{l_2-1} \ldots x_n^{l_n-1} \, dx_1, dx_2, \ldots dx_n$$

$$= \frac{\Gamma(l_1)\Gamma(l_2)\ldots\Gamma(l_n)}{\Gamma(1+l_1+l_2+\ldots+l_n)} \quad \ldots(2)$$

with condition $x_1 + x_2 + \ldots + x_n \leq 1$.

If the condition $x_1 + x_2 + \ldots + x_n \leq h$, then putting

$$\frac{x_1}{h} = u_1, \frac{x_2}{h} = u_2 \ldots \frac{x_n}{h} = u_n \text{ so that}$$

We have $$\iint \ldots \int x_1^{l_1-1} x_2^{l_2-1} \ldots x_n^{l_n-1} \, dx_1 \, dx_2 \ldots dx_n$$

$$= h^{l_1+l_2+\ldots+l_n} \iint \ldots \int u_1^{l_1-1} u_2^{l_2-1} \ldots u_n^{l_n-1} \, du_1 \, du_2 \ldots du_n$$

Subject to the condition $u_1 + u_2 + \ldots + u_n \leq 1$

$$=h^{l_1+l_2+\ldots+l_n}\frac{\Gamma(l_1)\Gamma(l_2)\ldots\Gamma(l_n)}{\Gamma(1+l_1+l_2+\ldots+l_n)}. \qquad \ldots(3)$$

(Using the assumed result (2)).

Now for $n+1$ variables the conditions are

$$x_1+x_2+\ldots+x_n+x_{n+1}\le 1$$

i.e., $\quad x_2+x_3+\ldots+x_n+x_{n+1}\le 1-x_1$

and $\quad 0\le x_1\le 1.$

We have $\quad \iint\ldots\int x_1^{l_1-1}\,x_2^{l_2-1}\ldots x_n^{l_n-1}\,x_{n+1}^{l_{n+1}-1}\,dx_1\,dx_2\ldots dx_n\,dx_{n+1}$

where $x_1+x_2+\ldots+x_{n+1}\le 1$

$$=\int_{x_1=0}^{1}x_1^{l_1-1}\iint\ldots\left[\int x_2^{l_2-1}\ldots x_{n+1}^{l_{n+1}-1}\,dx_2\ldots dx_{n+1}\right]dx_1$$

Using (3)

$$=\frac{\Gamma(l_2)\Gamma(l_3)\ldots\Gamma(l_{n+1})}{\Gamma(l_1+1+l_2+\ldots+l_n+l_{n+1})}\cdot\int_0^1 x_1^{l_1-1}(1-x_1)^{(1+l_2+l_3+\ldots+l_{n+1})+1}\,dx_1$$

$$=\frac{\Gamma(l_2)\Gamma(l_3)\ldots\Gamma(l_{n+1})}{\Gamma(1+l_2+\ldots+l_{n+1}l_{n+1})}\cdot\frac{\Gamma(l_1)\Gamma(1+l_2+\ldots+l_{n+1})}{\Gamma(1+l_1+l_2+\ldots+l_n+l_{n+1})}$$

$$=\frac{\Gamma(l_1)\Gamma(l_2)\ldots\Gamma(l_{n+1})}{\Gamma(1+l_1+l_2+\ldots+l_{n+1})}. \qquad \ldots(4)$$

The result (4) shows that the theorem hold for $(n+1)$ variables. Hence, by principle of mathematical induction, theorem is true for all values of n.

Q.10. State and prove Liouvilie's extension of Dirichlet's theorem.

Ans. Liouvillie's extension of Dirichlet's theorem : If the variables x, y, z are all positive such that

$$h_1\le x+y+z\le h_2,$$

then the triple integral

$$\iiint f(x+y+z)\,x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz$$

$$=\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)}\int_{h_1}^{h_2}f(u)\,u^{l+m+n-1}\,du.$$

Proof : Let $I=\iiint x^{l-1}\,y^{m-1}\,z^{n-1}\,dx\,dy\,dz$, integrated over some region.

Subject to the condition $x+y+z\le u$, we have by Dirichlet's theorem

$$I=u^{l+m+n}\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)} \qquad \ldots(1)$$

If the condition be $x+y+z\le u+\delta u$, then

$$I=(u+\delta u)^{l+m+n}\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)} \qquad \ldots(2)$$

Therefore the value of the integral I extended to all such positive values of the variables as make the sum of the variables lie between u and $u+\delta u$ is

$$=\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}[(u+\delta u)^{l+m+n}-u^{l+m+n}],$$

[subtracting (2) from (1)]

$$=\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}u^{l+m+n}\left[\left(1+\frac{\delta u}{u}\right)^{l+m+n}-1\right]$$

$$=\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}u^{l+m+n}\left[1+(l+m+n)\frac{\delta u}{u}+\ldots-1\right]$$

$$=\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}(l+m+n)u^{l+m+n-1}\,\delta u,$$

to the first order of approximation

$$=\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)}u^{l+m+n-1}\,\delta u.$$

Now consider the integral

$$\iiint f(x+y+z)\,x^{l-1}\,y^{m-1}\,z^{n-1}\,dx\,dy\,dz,$$

subject to the condition $h_1 \le x+y+z \le h_2$.

If $x+y+z$ lies between u and $u+\delta u$, the value of $f(x+y+z)$ can only differ from $f(u)$ by a small quantity of the same order as δu. Hence neglecting square of δu, the part of the integral

$$\iiint f(x+y+z)\,x^{l-1}\,y^{m-1}\,z^{n-1}\,dx\,dy\,dz$$

which arises from supposing the sum of the variables to lie between u and $u+\delta u$ is ultimately equal to $\dfrac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)}f(u).u^{l+m+n-1}\,\delta u.$

Therefore the whole integral

$$\iiint f(x+y+z)\,x^{l-1}\,y^{m-1}\,z^{n-1}\,dx\,dy\,dz,$$

where $h_1 \le x+y+z \le h_2$, is equal to

$$\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)}\int_{h_1}^{h_2} f(u).u^{l+m+n-1}\,du.$$

❑

UNIT VIII

SECTION-A VERY SHORT ANSWER TYPE QUESTIONS

Q.1. Define scalar fields.

Ans. A scalar point function f defined over some region R such that to each point $P(x, y, z)$ in space, there corresponds a unique scalar $f(P)$, is called a scalar field.

Q.2. If $\frac{d\mathbf{u}}{dt} = \mathbf{w}\times\mathbf{u}$, $\frac{d\mathbf{v}}{dt} = \mathbf{w}\times\mathbf{v}$, show that $\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \mathbf{w}\times(\mathbf{u}\times\mathbf{v})$.

Ans. We have

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \frac{d\mathbf{u}}{dt}\times\mathbf{v} + \mathbf{u}\times\frac{d\mathbf{v}}{dt} = (\mathbf{w}\times\mathbf{u})\times\mathbf{v} + \mathbf{u}\times(\mathbf{w}\times\mathbf{v})$$

$$= (\mathbf{v}\bullet\mathbf{w})\,\mathbf{u} - (\mathbf{v}\bullet\mathbf{u})\,\mathbf{w} + (\mathbf{u}\bullet\mathbf{v})\,\mathbf{w} - (\mathbf{u}\bullet\mathbf{w})\,\mathbf{v}$$

$$= (\mathbf{v}\bullet\mathbf{w})\,\mathbf{u} - (\mathbf{u}\bullet\mathbf{w})\,\mathbf{v} \qquad [\because\ \mathbf{u}\bullet\mathbf{v} = \mathbf{v}\bullet\mathbf{u}]$$

$$= (\mathbf{w}\bullet\mathbf{v})\,\mathbf{u} - (\mathbf{w}\bullet\mathbf{u})\,\mathbf{v} = \mathbf{w}\times(\mathbf{u}\times\mathbf{v}).$$

Q.3. Show that : $\frac{d^2}{dt^2}\left(\boldsymbol{r}\times\frac{d\boldsymbol{r}}{dt}\right) = \frac{d\boldsymbol{r}}{dt}\times\frac{d^2\boldsymbol{r}}{dt^2} + \boldsymbol{r}\times\frac{d^3\boldsymbol{r}}{dt^3}$

Ans.

$$\frac{d}{dt}\left(\boldsymbol{r}\times\frac{d\boldsymbol{r}}{dt}\right) = \frac{d\boldsymbol{r}}{dt}\times\frac{d\boldsymbol{r}}{dt} + \boldsymbol{r}\times\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{r}\times\frac{d^2\boldsymbol{r}}{dt^2} \qquad \left[\because\ \frac{d\boldsymbol{r}}{dt}\times\frac{d\boldsymbol{r}}{dt} = 0\right]$$

$$\therefore \quad \frac{d^2}{dt^2}\left(\boldsymbol{r}\times\frac{d\boldsymbol{r}}{dt}\right) = \frac{d}{dt}\left(\boldsymbol{r}\times\frac{d^2\boldsymbol{r}}{dt^2}\right) = \frac{d\boldsymbol{r}}{dt}\times\frac{d^2\boldsymbol{r}}{dt^2} + \boldsymbol{r}\times\frac{d^3\boldsymbol{r}}{dt^3}$$

Q.4. If $f(x, y, z) = 3x^2y - y^3z^2$, find grad f and |grad f| at $(1, -2, -1)$.

Ans. Since we know that

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}(3x^2y - y^3z^2)\hat{i} + \frac{\partial}{\partial y}(3x^2y - y^3z^2)\hat{j} + \frac{\partial}{\partial z}(3x^2y - y^3z^2)\hat{k}$$

$$= 6xy\,\hat{i} + (3x^2 - 3y^2z^2)\,\hat{j} + (-2y^3z)\,\hat{k}$$

At $(1, -2, -1)$ $\quad \text{grad } f = -12\hat{i} - 9\hat{j} - 16\hat{k}$

and $\quad |\text{grad } f| = \sqrt{144 + 81 + 256} = \sqrt{481}.$

Q.5. In what direction from the point (1, 1, –1) is the directional derivative of $f = x^2 - 2y^2 + 4z^2$ a maximum? Also find the value of this maximum directional derivative.

Ans. We have grad $f = 2x\,\mathbf{i} - 4y\,\mathbf{j} + 8z\,\mathbf{k} = 2\,\mathbf{i} - 4\,\mathbf{j} - 8\,\mathbf{k}$ at the point (1, 1, –1).

The directional derivative of f is maximum in the direction of grad $f = 2\,\mathbf{i} - 4\,\mathbf{j} - 8\,\mathbf{k}$.

The maximum value of this directional derivative

$$=|\text{grad } f|=|2\,\mathbf{i}-4\,\mathbf{j}-8\,\mathbf{k}|=\sqrt{(4+16+64)}=\sqrt{(84)}=2\sqrt{(21)}.$$

Q.6. What is the greatest rate of increase of $u=xyz^2$ at the point $(1,0,3)$?

Ans. We have $\nabla u = yz^2\ \mathbf{i}+xz^2\ \mathbf{j}+2xyz\ \mathbf{k}$.

$\therefore$ at the point (1, 0, 3), we have $\nabla u = 0\,\mathbf{i}+9\,\mathbf{j}+0\,\mathbf{k}=9\,\mathbf{j}$.

The greatest rate of increase of u at the point (1, 0, 3)

$$=\text{the maximum value of } \frac{du}{ds} \text{ at the point } (1,0,3)$$

$$=|\nabla u|, \text{ at the point } (1,0,3)=|9\,\mathbf{j}|=9.$$

Q.7. Prove that curl $\mathbf{r}=\mathbf{0}$.

Ans. We have by definition

$$\text{curl } \mathbf{r}=\nabla\times\mathbf{r}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times\mathbf{r}=\mathbf{i}\times\frac{\partial \mathbf{r}}{\partial x}+\mathbf{j}\times\frac{\partial \mathbf{r}}{\partial y}+\mathbf{k}\times\frac{\partial \mathbf{r}}{\partial z}.$$

Now $\quad \mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$.

$$\therefore \quad \frac{\partial \mathbf{r}}{\partial x}=\mathbf{i}, \frac{\partial \mathbf{r}}{\partial y}=\mathbf{j}, \frac{\partial \mathbf{r}}{\partial z}=\mathbf{k}.$$

$$\therefore \quad \text{curl } \mathbf{r}=\mathbf{i}\times\mathbf{i}+\mathbf{j}\times\mathbf{j}+\mathbf{k}\times\mathbf{k}=\mathbf{0}+\mathbf{0}+\mathbf{0}=\mathbf{0}.$$

Q.8. Define gradient of a scalar field.

Ans. Gradient of a scalar field : Let $f(x,y,z)$ be a scalar point function which is defined over some region R in space and also differentiable at each point (x,y,z) in R, then the gradient of $f(x,y,z)$ is defined as

$$\text{grad } f=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}$$

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or
$$\text{grad } f=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)f=\nabla f$$

Thus gradient of f can also be written in terms of vector differential operator (∇). Since ∇ is a vector quantity, thus ∇f is a vector whose components are $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$. Hence, gradient of a scalar field is a vector field.

Q.9. Determine the constant a so that the vector $\mathbf{V}=(x+3y)\mathbf{i}+(y-2z)\mathbf{j}+(x+ax)\mathbf{k}$ is solenoidal.

Ans. A vector **V** is said to be solenoidal if div $\mathbf{V}=0$.

We have $\quad \text{div } \mathbf{V}=\nabla\bullet\mathbf{V}=\frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-2z)+\frac{\partial}{\partial z}(x+az)=1+1+a=2+a.$

Now $\quad$ div $\mathbf{V}=0$ if $2+a=0$ *i.e.,* if $a=-2$.

Q.10. If V is a constant vector, show that
(i) div V = 0, (ii) curl V = 0.

Ans. (i) We have $\text{div}\,\mathbf{V}=\mathbf{i}\bullet\frac{\partial\mathbf{V}}{\partial x}+\mathbf{j}\bullet\frac{\partial\mathbf{V}}{\partial y}+\mathbf{k}\bullet\frac{\partial\mathbf{V}}{\partial z}=\mathbf{i}\bullet\mathbf{0}+\mathbf{j}\bullet\mathbf{0}+\mathbf{k}\bullet\mathbf{0}=0.$

(ii) We have $\text{curl}\,\mathbf{V}=\mathbf{i}\times\frac{\partial\mathbf{V}}{\partial x}+\mathbf{j}\times\frac{\partial\mathbf{V}}{\partial y}+\mathbf{k}\times\frac{\partial\mathbf{V}}{\partial z}=\mathbf{i}\times\mathbf{0}+\mathbf{j}\times\mathbf{0}+\mathbf{k}\times\mathbf{0}=0.$

Q.11. Using Gauss's divergence theorem evaluate

$$\iint_S [(x+z)\,dy\,dz+(y+z)\,dz\,dx+(x+y)\,dx\,dy]$$

where S is the surface of the sphere $x^2+y^2+z^2=4$.

Ans. By Gauss's divergence theorem, we have

$$=\iint_S [(x+z)\,dy\,dz+(y+z)\,dz\,dx+(x+y)\,dx\,dy]$$

$$=\iiint_V\left[\frac{\partial}{\partial x}(x+z)+\frac{\partial}{\partial y}(y+z)+\frac{\partial}{\partial z}(x+y)\right]dx\,dy\,dz=\iiint_V 2\,dx\,dy\,dz$$

$$=2\iiint_V dV, \text{ where } V \text{ is the volume of the sphere } x^2+y^2+z^2=4$$

$$=2\left[\frac{4}{3}\pi(2)^3\right]=\frac{64}{3}\pi.$$

Q.12. Prove that $\oint_C \boldsymbol{r}\,.\,d\boldsymbol{r}=0$

Ans. Using Stoke's theorem, we have

$$\oint_C \boldsymbol{r}\,.\,d\boldsymbol{r}=\iint_S(\nabla\times\boldsymbol{r})\,.\,\boldsymbol{n}\,dS=0 \qquad [\because\ \nabla\times\boldsymbol{r}=0]$$

SECTION-B SHORT ANSWER TYPE QUESTIONS

Q.1. If $\boldsymbol{r}=\boldsymbol{a}\sin\omega t+\boldsymbol{b}\cos\omega t+\frac{\boldsymbol{c}t}{\omega^2}\sin\omega t$, prove that $\frac{d^2\boldsymbol{r}}{dt^2}+\omega^2\boldsymbol{r}=\frac{2\boldsymbol{c}}{\omega}\cos\omega t$.

where *a, b, c* are constant vectors and ω is a constant scalar.

Ans. Since ***a, b, c*** are constant vectors so $\frac{d\boldsymbol{a}}{dt}=0, \frac{d\boldsymbol{b}}{dt}=0$ and $\frac{d\boldsymbol{c}}{dt}=0$

and $$\boldsymbol{r}=\boldsymbol{a}\sin\omega t+\boldsymbol{b}\cos\omega t+\frac{\boldsymbol{c}t}{\omega^2}\sin\omega t \qquad ...(1)$$

$$\therefore \quad \frac{d\boldsymbol{r}}{dt}=\omega\boldsymbol{a}\cos\omega t-\omega\boldsymbol{b}\sin\omega t+\frac{\boldsymbol{c}}{\omega^2}\sin\omega t+\frac{\boldsymbol{c}t}{\omega}\cos\omega t$$

and $$\frac{d^2\boldsymbol{r}}{dr^2}=\frac{d}{dt}\left(\frac{d\boldsymbol{r}}{dt}\right)=-\omega^2\boldsymbol{a}\sin\omega t-\omega^2\boldsymbol{b}\cos\omega t+\frac{\boldsymbol{c}}{\omega}\cos\omega t+\frac{\boldsymbol{c}}{\omega}\cos\omega t-\boldsymbol{c}t\sin\omega t$$

$$=-\omega^2\left(\boldsymbol{a}\sin\omega t+\boldsymbol{b}\cos\omega t+\frac{\boldsymbol{c}t}{\omega^2}\sin\omega t\right)+\frac{2\boldsymbol{c}}{\omega}\cos\omega t=-\omega^2\boldsymbol{r}+\frac{2\boldsymbol{c}}{\omega}\cos\omega t.$$

$$\therefore \quad \frac{d^2\boldsymbol{r}}{dt^2}+\omega^2\boldsymbol{r}=\frac{2\boldsymbol{c}}{\omega}\cos\omega t.$$

Q.2. If $\mathbf{a} = \sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} + \theta\mathbf{k}$, $\mathbf{b} = \cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j} - 3\mathbf{k}$, and $\mathbf{c} = 2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, find $\frac{d}{d\theta}\{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\}$ at $\theta = \frac{\pi}{2}$.

Ans. We have $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & -\sin\theta & -3 \\ 2 & 3 & -3 \end{vmatrix}$

$$= (3\sin\theta + 9)\,\mathbf{i} + (3\cos\theta - 6)\,\mathbf{j} + (3\cos\theta + 2\sin\theta)\,\mathbf{k}.$$

$$\therefore \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin\theta & \cos\theta & \theta \\ 2\sin\theta + 9 & 3\cos\theta - 6 & 3\cos\theta + 2\sin\theta \end{vmatrix}$$

$$= (3\cos^2\theta + 2\sin\theta\cos\theta - 3\theta\cos\theta + 6\theta)\,\mathbf{i}$$
$$+ (3\theta\sin\theta + 9\theta - 3\sin\theta\cos\theta - 2\sin^2\theta)\,\mathbf{j} + (-6\sin\theta - 9\cos\theta)\,\mathbf{k}.$$

$$\therefore \quad \frac{d}{d\theta}\{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\} = (-6\cos\theta\sin\theta + 2\cos^2\theta - 2\sin^2\theta$$
$$-3\cos\theta + 3\theta\sin\theta + 6)\,\mathbf{i} + (3\sin\theta + 3\theta\cos\theta + 9 - 3\cos^2\theta + 3\sin^2\theta$$
$$-4\sin\theta\cos\theta)\,\mathbf{j} + (-6\cos\theta + 9\sin\theta)\,\mathbf{k}.$$

Putting $\theta = \pi/2$, we get the required derivative $= \left(4 + \frac{3}{2}\pi\right)\mathbf{i} + 15\mathbf{j} + 9\mathbf{k}$.

Q.3. If $\boldsymbol{r} = (\cos nt)\,\boldsymbol{i} + (\sin nt)\,\boldsymbol{j}$, where n is a constant and t varies, show that $\boldsymbol{r} \times \frac{d\mathbf{r}}{dt} = n\hat{k}$.

Ans. Since $\hat{i}$ and $\hat{j}$ are constant vectors so $\frac{d\hat{i}}{dt} = \mathbf{0}$, $\frac{d\hat{j}}{dt} = \mathbf{0}$ and

$$\boldsymbol{r} = (\cos nt)\,\hat{i} + (\sin nt)\,\hat{j}. \qquad \text{...(1)}$$

Differentiating (1) w.r.t. 't', we get

$$\frac{d\boldsymbol{r}}{dt} = -n\,(\sin nt)\,\hat{i} + n\,(\cos nt)\,\hat{j}. \qquad \text{...(2)}$$

Now $\boldsymbol{r} \times \frac{d\boldsymbol{r}}{dt} = \boldsymbol{r} \times [-n\,(\sin nt)\,\hat{i} + n\,(\cos nt)\,\hat{j}]$

$$= [(\cos nt)\,\hat{i} + (\sin nt)\,\hat{j}] \times [-n\,(\sin nt)\,\hat{i} + n\,(\cos nt)\,\hat{j}] \qquad \text{[From (1)]}$$
$$= (n\cos^2 nt)\,\hat{i} \times \hat{j} - n\,(\sin^2 nt)\,\hat{j} \times \hat{i}$$
$$= n\,(\cos^2 nt)\,\hat{k} + n\,(\sin^2 nt)\,\hat{k} \qquad [\because\ \hat{j} \times i = -\hat{k} \text{ and } \hat{i} \times \hat{j} = \hat{k}]$$
$$= (\cos^2 nt + \sin^2 nt)\,n\,\hat{k} = n\,\hat{k} \qquad (\because\ \cos^2 nt + \sin^2 nt = 1)$$

Hence, $\boldsymbol{r} \times \frac{d\boldsymbol{r}}{dt} = r\,\hat{k}$.

Q.4. Prove that

(i) $\nabla(\mathbf{r}.\mathbf{a})=\mathbf{a}$ (ii) $\nabla[\mathbf{r}\ \mathbf{a}\ \mathbf{b}]=\mathbf{a}\times\mathbf{b}$,

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

Ans. Suppose $\mathbf{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\mathbf{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $\mathbf{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$,

then $$\mathbf{r}.\mathbf{a}=xa_1+a_2y+a_3z$$

and $$\mathbf{r}.(\mathbf{a}\times\mathbf{b})=\begin{vmatrix} x & y & z \\ a_1 & a_2 & a_3 \\ b_1 & a_2 & b_3 \end{vmatrix}$$

$$=x(a_2b_3-a_3b_2)+y(a_3b_1-a_1b_3)+z(a_1b_2-a_2b_1)$$

(i) $\nabla(\mathbf{r}.\mathbf{a})=\nabla(xa_1+a_2y+a_3z)=a_1\nabla(x)+a_2\nabla(y)+a_3\nabla(z)$

$=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ $\quad(\because \nabla(x)=\hat{i},\nabla(y)=\hat{j},\nabla(z)=\hat{k})$

$=\mathbf{a}.$

(ii) $\nabla[\mathbf{r}\ \mathbf{a}\ \mathbf{b}]=\nabla(\mathbf{r}.(\mathbf{a}\times\mathbf{b}))$

$=\nabla[x(a_2b_3-a_3b_2)+y(a_3b_1-a_1b_3)+z(a_1b_2-a_2b_1)]$

$=\nabla[x(a_2b_3-a_3b_2)+\nabla[y(a_3b_1-a_1b_3)]+\nabla[z(a_1b_2-a_2b_1)]$

$=(a_2b_3-a_3b_2)\nabla(x)+(a_3b_1-a_1b_3)\nabla(y)+(a_1b_2-a_2b_1)\nabla(z)$

$=(a_2b_3-a_3b_2)\hat{i}+(a_3b_1-a_1b_3)\hat{j}+(a_1b_2-a_2b_1)\hat{k}$

$$=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & a_2 & b_3 \end{vmatrix}=\mathbf{a}\times\mathbf{b}$$

$\therefore\ \nabla[\mathbf{r}\ \mathbf{a}\ \mathbf{b}]=\mathbf{a}\times\mathbf{b}.$

Q.5. If $f(x,y,z)=3x^2y-y^3z^2$, find grad f at the point $(1,-2,-1)$.

Ans. We have

$$\text{grad } f=\nabla f=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)(3x^2y-y^3z^2)$$

$$=\mathbf{i}\frac{\partial}{\partial x}(3x^2y-y^3z^2)+\mathbf{j}\frac{\partial}{\partial y}(3x^2y-y^3z^2)+\mathbf{k}\frac{\partial}{\partial z}(3x^2y-y^3z^2)$$

$$=\mathbf{i}(6xy)+\mathbf{j}(3x^2-3y^2z^2)+\mathbf{k}(-2y^3z)$$

$$=6xy\,\mathbf{i}+(3x^2-3y^2z^2)\mathbf{j}-2y^3z\,\mathbf{k}.$$

Putting $x=1, y=-2, z=-1$, we get

$\nabla f=6(1)(-2)\mathbf{i}+\{3(1)^2-3(-2)^2(-1)^2\}\mathbf{j}-2(-2)^3(-1)\mathbf{k}=-12\mathbf{i}-9\mathbf{j}-16\mathbf{k}.$

Q.6. If $\mathbf{A} = x^2 yz\,\mathbf{i} - 2xz^3\,\mathbf{j} + xz^2\,\mathbf{k}, \mathbf{B} = 2z\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$, find the value of $\frac{\partial^2}{\partial x\,\partial y}(\mathbf{A}\times\mathbf{B})$ at $(1, 0, -2)$.

Ans. We have $\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x^2 yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$

$$= (2x^3z^3 - xyz^2)\,\mathbf{i} + (2xz^3 + x^4 yz)\,\mathbf{j} + (x^2y^2z + 4xz^4)\,\mathbf{k}.$$

$$\therefore \quad \frac{\partial}{\partial y}(\mathbf{A}\times\mathbf{B}) = -xz^2\mathbf{i} + x^4 z\,\mathbf{j} + 2x^2 yz\mathbf{k}.$$

Again $\frac{\partial^2}{\partial x\,\partial y}(\mathbf{A}\times\mathbf{B}) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}(\mathbf{A}\times\mathbf{B})\right\} = -z^2\mathbf{i} + 4x^3 z\,\mathbf{j} + 4xyz\,\mathbf{k}.$ ...(1)

Putting $x = 1, y = 0$ and $z = -2$ in (1), we get the required derivative at the point $(1, 0, -2) = -4\mathbf{i} - 8\mathbf{j}$.

Q.7. If $\mathbf{f} = x^2 y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$, find

(i) div f, (ii) curl f, (iii) curl curl f.

Ans. (i) We have

$$\text{div } \mathbf{f} = \nabla \bullet \mathbf{f} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet (x^2 y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k})$$

$$= \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(2yz) = 2xy + 0 + 2y = 2y(x+1).$$

(ii) We have $\text{curl } \mathbf{f} = \nabla\times\mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -2xz & 2yz \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2 y)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2 y)\right]\mathbf{k}$$

$$= (2z + 2x)\,\mathbf{i} - 0\,\mathbf{j} + (-2z - x^2)\,\mathbf{k} = (2x + 2z)\,\mathbf{i} - (x^2 + 2z)\,\mathbf{k}.$$

(iii) We have $\text{curl curl } \mathbf{f} = \nabla\times(\nabla\times\mathbf{f}) = \nabla\times[(2x+2z)\,\mathbf{i} - (x^2+2z)\,\mathbf{k}]$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+2z & 0 & -x^2-2z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-x^2 - 2z)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-x^2 - 2z) - \frac{\partial}{\partial z}(2x + 2z)\right]\mathbf{j} + \left[0 - \frac{\partial}{\partial y}(2x + 2z)\right]\mathbf{k}$$

$$= 0\,\mathbf{i} - (-2x - 2)\,\mathbf{j} + (0 - 0)\,\mathbf{k} = (2x + 2)\,\mathbf{j}.$$

Q.8. Prove that div grad $(r^n) = n(n+1)r^{n-2}$.

Ans. Since we have grad $(r^n) = (nr^{n-1} \text{ grad } r)$ $\quad [\because \text{grad } f(r) = f'(r)(\nabla r)]$

$$= nr^{n-1}\frac{\mathbf{r}}{r} \qquad \left[\because \text{grad } r = \frac{\mathbf{r}}{r}\right]$$

$$= nr^{n-2}\,\mathbf{r}.$$

Now div grad $(r^n) = \text{div}\,(nr^{n-2}\,\mathbf{r})$

$$= n\,\text{div}\,(r^{n-2}\,\mathbf{r}) = n[r^{n-2}\,\nabla.\mathbf{r} + \mathbf{r}.\text{grad}\,(r^{n-2})]$$

$$= n[3r^{n-2} + \mathbf{r}.((n-2)r^{n-3}\text{ grad } r)] \qquad (\because \nabla.\mathbf{r} = 3)$$

$$= n\left[3r^{n-2} + \mathbf{r}.\left((n-2)r^{n-3}\frac{\mathbf{r}}{r}\right)\right]$$

$$= n[3r^{n-2} + (n-2)r^{n-4}\,\mathbf{r}.\mathbf{r}]$$

$$= n[3r^{n-2} + (n-2)r^{n-4}\,r^2] \qquad [\because \mathbf{r}.\mathbf{r} = r^2]$$

$$= n(n+1)r^{n-2}$$

$\therefore$ div grad $r^n = n(n+1)r^{n-2}$

Q.9. Prove that div $\hat{\mathbf{r}} = 2/r$.

Ans. $\text{div}\,(\hat{\mathbf{r}}) = \text{div}\left(\frac{1}{r}\mathbf{r}\right)$.

Alternate Method :

$$\text{div}\,\hat{\mathbf{r}} = \text{div}\left(\frac{1}{r}\mathbf{r}\right) = \text{div}\left[\frac{1}{r}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\right]$$

$$= \text{div}\left(\frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \left(\frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x}\right) + \left(\frac{1}{r} - \frac{y}{r^2}\frac{\partial r}{\partial y}\right) + \left(\frac{1}{r} - \frac{z}{r^2}\frac{\partial r}{\partial z}\right).$$

Now $r^2 = x^2 + y^2 + z^2$. $\quad \therefore\ 2r\frac{\partial r}{\partial x} = 2x \quad$ *i.e.,* $\frac{\partial r}{\partial x} = \frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$$\therefore \quad \text{div}\,\hat{\mathbf{r}} = \frac{3}{r} - \left(\frac{x}{r^2}\frac{x}{r} + \frac{y}{r^2}\frac{y}{r} + \frac{z}{r^2}\frac{z}{r}\right) = \frac{3}{r} - \frac{x^2+y^2+z^2}{r^3} = \frac{3}{r} - \frac{r^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}.$$

Q.10. Prove that curl $[\mathbf{r}\times(\mathbf{a}\times\mathbf{r})] = 3\mathbf{r}\times\mathbf{a}$, where a is a constant vector.

Ans. curl $[\mathbf{r}\times(\mathbf{a}\times\mathbf{r})]$

$$= \nabla\times[(\mathbf{r}\bullet\mathbf{r})\mathbf{a} - (\mathbf{r}\bullet\mathbf{a})\mathbf{r}] \qquad [\because \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\bullet\mathbf{c})\mathbf{b} - (\mathbf{a}\bullet\mathbf{b})\mathbf{c}]$$

$$= \nabla\times[r^2\mathbf{a} - (\mathbf{r}\bullet\mathbf{a})\mathbf{r}] \qquad [\because \mathbf{r}\bullet\mathbf{r} = \mathbf{r}^2 = r^2]$$

$= \nabla \times (r^2 \mathbf{a}) - \nabla \times [(\mathbf{r} \bullet \mathbf{a})\mathbf{r}]$ $\quad [\because \nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}]$

$= (\nabla r^2) \times \mathbf{a} + r^2 (\nabla \times \mathbf{a}) - [\nabla (\mathbf{r} \bullet \mathbf{a})] \times \mathbf{r} - (\mathbf{r} \bullet \mathbf{a})(\nabla \times \mathbf{r})$

$[\because \nabla \times (\phi \mathbf{A}) = (\nabla \phi) \times \mathbf{A} + \phi (\nabla \times \mathbf{A})]$

$= (2r \nabla r) \times \mathbf{a} + r^2 \mathbf{0} - [\nabla (\mathbf{r} \bullet \mathbf{a})] \times \mathbf{r} - (\mathbf{r} \bullet \mathbf{a}) \mathbf{0}$ $\quad [\because \nabla f(r) = f'(r) \nabla r; \nabla \times \mathbf{a} = \mathbf{0}, \mathbf{a}$ being a constant vector; and $\nabla \times \mathbf{r} = \mathbf{0}]$

$= \left(2r \dfrac{1}{r} \mathbf{r}\right) \times \mathbf{a} - [\nabla (\mathbf{r} \bullet \mathbf{a})] \times \mathbf{r}$

$= 2\mathbf{r} \times \mathbf{a} - \mathbf{a} \times \mathbf{r}$ $\quad [\because \nabla (\mathbf{r} \bullet \mathbf{a}) = \mathbf{a}$, if $\mathbf{a}$ is a constant vector]

$= 2\mathbf{r} \times \mathbf{a} + \mathbf{r} \times \mathbf{a} = 3\mathbf{r} \times \mathbf{a}.$

Q.11. Find the angle between the surface $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

Ans. Let the given surfaces be

$f_1(x, y, z) \equiv x^2 + y^2 + z^2 = 9$ as $f_1(x, y, z) = c_1$ ...(1)

$f_2(x, y, z) \equiv x^2 + y^2 - z = 3$ as $f_2(x, y, z) = c_2$ ...(2)

Normal vector to surface (1) is

$$n_1 = \text{grad } f_1 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

At point $(2, -1, 2)$,

$$n_1 = 2.2\hat{i} + 2(-1)\hat{j} + 2.2\hat{k} = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

Normal vector to surface (2) is

$$n_2 = \text{grad } f_2 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 - z^2) = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

At point $(2, -1, 2)$,

$$n_2 = 2.2\hat{i} + 2(-1)\hat{j} - \hat{k} = 4\hat{i} - 2\hat{j} - \hat{k}.$$

Now let θ be the angle between surfaces (1) and (2), then the angle between their normals n_1 and n_2 is also θ.

$$\therefore \quad \cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{4.4 + (-2)(-2) + 4(-1)}{\sqrt{(4)^2 + (-2)^2 + (4)^2}\sqrt{(4)^2 + (-2)^2 + (-1)^2}}$$

$$= \frac{16 + 4 - 4}{\sqrt{36}\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

Q.12. Find the equation of the tangent plane to the surface $yz - zx + xy + 5 = 0$ at the point $(1, -1, 2)$.

Ans. Let $f = yz - zx + xy + 5 = 0$, then

$$\frac{\partial f}{\partial x} = -z + y, \frac{\partial f}{\partial y} = z + x, \frac{\partial f}{\partial z} = y - x.$$

$\therefore \qquad \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} + (y-z)\hat{i} + (z+x)\hat{j} + (y-x)\hat{k}$

At $(1,-1,2)$ $\nabla f = -3\hat{i} + 3\hat{j} - 2\hat{k}$.

Let $Q(X, Y, Z)$ be any point on the tangent plane to the surface and P is given as $(1,-1,2)$.

$\therefore \qquad \overrightarrow{PQ} = (X-1)\hat{i} + (Y+1)\hat{j} + (Z-2)\hat{k}$

For the equation of tangent plane at $(1,-1,2)$, we have

$$\nabla f \,.\, \overrightarrow{PQ} = 0$$

$$(X-1)(-3) + (Y+1)3 + (Z-2)(-2) = 0$$

$$-3X + 3Y - 2Z + 3 + 3 + 4 = 0$$

or $\qquad 3X - 3Y + 2Z = 10$ or $3x - 3y + 2z = 10$

Q.13. Find a unit normal vector to the level surface $x^2 y + 2xy = 4$ at the point $(2,-2,3)$.

Ans. The equation of the level surface is $f(x, y, z) \equiv x^2 y + 2xz = 4$.

The vector grad f is along the normal to the surface at the point (x, y, z).

We have $\qquad$ grad $f = \nabla(x^2 y + 2xz) = (2xy + 2z)\,\mathbf{i} + x^2\mathbf{j} + 2x\,\mathbf{k}$.

$\therefore$ at the point $(2,-2,3)$, grad $f = -2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}$.

$\therefore$ $-2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}$ is a vector along the normal to the given surface at the point $(2,-2,3)$. Hence a unit normal vector to the surface at this point.

$$= \frac{-2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}}{|-2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}|} = \frac{-2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}}{\sqrt{(4+16+16)}} = -\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

The vector $-\left(-\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$ *i.e.,* $\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$ is also a unit normal vector to the given surface at the point $(2,-2,3)$.

Q.14. Find the directional derivative of $f(x, y, z) = x^2 yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$.

Ans. Let $\boldsymbol{a} = 2\hat{i} - \hat{j} - 2\hat{k}$, then

$$\hat{a} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{(4+1+4)}} = \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k}).$$

Since $f(x, y, z) = x^2 yz + 4xz^2$.

$\therefore \qquad \frac{\partial f}{\partial x} = 2xyz + 4z^2$

$\qquad \frac{\partial f}{\partial y} = x^2 z, \frac{\partial f}{\partial z} = x^2 y + 8xz.$

$\therefore \qquad \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = (2xyz + 4z^2)\hat{i} + x^2 z\,\hat{j} + (x^2 y + 8xz)\hat{k}$

At $(1, -2, -1)$

$$\nabla f = 8\hat{i} - \hat{j} - 10\hat{k}.$$

Now directional derivative of f at $(1, -2, -1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$ is

$$\nabla f \cdot \hat{a} = (8\hat{i} - \hat{j} - 10\hat{k}) \cdot \left(\frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})\right) = \frac{1}{3}(16 + 1 + 20) = \frac{37}{3}.$$

Q.15. Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point P (1, 2, 3) in the direction of the line PQ where Q is the point (5, 0, 4).

Ans. Here grad $f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$

$$= 2x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k} = 2\mathbf{i} - 4\mathbf{j} + 12\mathbf{k} \text{ at the point } (1, 2, 3).$$

Also $\vec{PQ}$ = position vector of Q − position vector of P

$$= (5\mathbf{i} + 0\mathbf{j} + 4\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

If $\hat{\mathbf{a}}$ be the unit vector in the direction of the vector $\vec{PQ}$, then

$$\hat{\mathbf{a}} = \frac{4\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}}{\sqrt{(16 + 4 + 1)}} = \frac{4\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}}{\sqrt{(21)}}.$$

∴ the required directional derivative

$$= (\text{grad } f) \bullet \hat{\mathbf{a}} = (2\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}) \bullet \left\{\frac{4\mathbf{i} - 2\mathbf{j} + \mathbf{k}}{\sqrt{(21)}}\right\}$$

$$= \frac{28}{\sqrt{(21)}} = \frac{28}{21}\sqrt{(21)} = \frac{4}{3}\sqrt{(21)}.$$

Q.16. If $F(t) = 3t^2\,\hat{i} + t\,\hat{j} + 2\,\hat{k}$ and $G(t) = 6t^2\,\hat{i} + (t-1)\,\hat{j} + 3t\,\hat{k}$ then find $\int_0^1 \left(\frac{dF}{dt} \cdot G + F \cdot \frac{dG}{dt}\right) dt$ and $\int_0^1 \left(F \times \frac{dG}{dt} + \frac{dF}{dt} \times G\right) dt$

Ans. We have $\mathbf{F} \cdot \mathbf{G} = 18t^4 + t(t-1) + 6t$

and

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3t^2 & t & 2 \\ 6t^2 & (t-1) & 3t \end{vmatrix}$$

$$= \hat{i}\,(3t^2 - 2t + 2) - \hat{j}\,(9t^3 - 12t^2) + \hat{k}\,(3t^3 - 3t - 6t^3)$$

$$= (3t^2 - 2t + 2)\,\hat{i} - (9t^3 - 12t^2)\,\hat{j} - (3t^3 - 3t)\,\hat{k}$$

Now, $\int_0^1 \left(\frac{d\mathbf{F}}{dt} \cdot \mathbf{G} + \mathbf{F} \cdot \frac{d\mathbf{G}}{dt}\right) dt = [\mathbf{F} \cdot \mathbf{G}]_0^1 = [18t^4 + t(t-1) + 6t]_0^1 = 18 + 6 = 24$

$$\int_0^1 \left(\mathbf{F} \times \frac{d\mathbf{G}}{dt} + \frac{d\mathbf{F}}{dt} \times \mathbf{G}\right) dt = [\mathbf{F} \times \mathbf{G}]_0^1$$

$$=[(3t^2-2t+2)\hat{i}-(9t^3-12t^2)\hat{j}-(3t^3+3t)\hat{k}]_0^1$$
$$=(3\hat{i}+3\hat{j}-6\hat{k})-(2i)=\hat{i}+3\hat{j}-6\hat{k}$$

Q.17. If $\mathbf{r}(t)=5t^2\mathbf{i}+t\mathbf{j}-t^3\mathbf{k}$, prove that $\int_1^2\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=-14\mathbf{i}+75\mathbf{j}-15\mathbf{k}$.

Ans. We have $\int\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\mathbf{r}\times\frac{d\mathbf{r}}{dt}+c$. $\therefore\ \int_1^2\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\left[\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right]_1^2$

Let us now find $\mathbf{r}\times\frac{d\mathbf{r}}{dt}$. We have $\frac{d\mathbf{r}}{dt}=10t\,\mathbf{i}+\mathbf{j}-3t^2\mathbf{k}$.

$$\therefore\quad \mathbf{r}\times\frac{d\mathbf{r}}{dt}=(5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\mathbf{k})\times(10t\,\mathbf{i}+\mathbf{j}-3t^2\,\mathbf{k})$$
$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5t^2 & t & -t^3\\ 10t & 1 & -3t^2\end{vmatrix}=-2t^3\,\mathbf{i}+5t^4\,\mathbf{j}-5t^2\,\mathbf{k}.$$

$$\therefore\quad \int_1^2\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\left[-2t^3\mathbf{i}+5t^4\mathbf{j}-5t^2\mathbf{k}\right]_1^2$$
$$=\left[-2t^3\right]_1^2\mathbf{i}+\left[5t^4\right]_1^2\mathbf{j}-\left[5t^2\right]_1^2\mathbf{k}=-14\mathbf{i}+75\mathbf{j}-15\mathbf{k}.$$

Q.18. Interpret the relations $\boldsymbol{r}\cdot\frac{d\boldsymbol{r}}{dx}=0$ and $\boldsymbol{r}\times\frac{d\boldsymbol{r}}{ds}=0$.

Ans. For $\quad \boldsymbol{r}\cdot\frac{d\boldsymbol{r}}{ds}=0 \Rightarrow 2\boldsymbol{r}\cdot\frac{d\boldsymbol{r}}{ds}=0.$

Integrating w.r.t. s, we get

$$\int\left(2\boldsymbol{r}\cdot\frac{d\boldsymbol{r}}{ds}\right)ds=\int 0\,ds$$

or $\quad \boldsymbol{r}^2=a$ (constant) $\Rightarrow$ $\boldsymbol{r}$ has constant magnitude.

Thus $\boldsymbol{r}$ describes a circle.

Again, for $\boldsymbol{r}\times\frac{d\boldsymbol{r}}{ds}=0 \Rightarrow \boldsymbol{r}$ and $\frac{d\boldsymbol{r}}{ds}$ are parallel.

Also $\frac{d\boldsymbol{r}}{ds}$ is a unit vector along tangent.

$\therefore$ $\boldsymbol{r}$ has constant direction that the tangent at every point is along $\boldsymbol{r}$. Thus $\boldsymbol{r}$ describes a straight line.

Q.19. By Gauss divergence theorem, show that

$$\iint_S (x^2\,\hat{i}+y^2\,\hat{j}+z^2\,\hat{k})\cdot\bar{n}\,dS=0,$$

where S is the surface of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Ans. By Gauss divergence theorem, we have

$$\iint_s (x^2\,\hat{i}+y^2\,\hat{j}+z^2\,\hat{k}).\bar{n}\,dS=\iiint_s (x^2\hat{i}+y^2\hat{j}+z^2\hat{k}).dV$$

where V is the volume enclosed by S

$$=\iiint_V (2x+2y+2z)\,dx\,dy\,dz=2\iiint_V (x+y+z)\,dx\,dy\,dz$$

$$=2\int_{z=-c}^{c}\int_{y=-b\sqrt{1-z^2/c^2}}^{y=b\sqrt{1-z^2/c^2}}\int_{y=-a\sqrt{1-y^2/b^2-z^2/c^2}}^{x=a\sqrt{1-y^2/b^2-z^2/c^2}}(x+y+z)\,dx\,dy\,dz$$

$$=2\int_{z=-c}^{c}\int_{y=-b\sqrt{1-z^2/c^2}}^{b\sqrt{1-z^2/c^2}}\left[\frac{x^2}{2}+x\,(y+z)\right]_{-a\sqrt{1-y^2/b^2-z^2/c^2}}^{a\sqrt{1-y^2/b^2-z^2/c^2}}dy\,dz$$

$$=4a\int_{z=-c}^{c}\int_{y=-b\sqrt{1-z^2/c^2}}^{y=b\sqrt{1-z^2/c^2}}(y+z)\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dy\,dz$$

$$=8a\int_{z=-c}^{c}\int_{0}^{b\sqrt{1-z^2/c^2}} z\left(\sqrt{\left(1-\frac{z^2}{c^2}\right)-\frac{y^2}{b^2}}\right)dy\,dz\ \text{[By the property of definite integral]}$$

$$=\frac{8a}{b}\int_{z=-c}^{c} z\left[\frac{y}{2}\sqrt{b^2\left(1-\frac{z^2}{c^2}\right)-y^2}+\frac{b^2}{2}\left(1-\frac{z^2}{c^2}\right)\sin^{-1}\left\{\frac{y}{b\sqrt{1-\frac{z^2}{c^2}}}\right\}\right]_0^{b\sqrt{1-z^2/c^2}}dz$$

$$=\frac{8a}{b}\int_{z=-c}^{c} z\left[\frac{b^2}{2}\left(1-\frac{z^2}{c^2}\right)\sin^{-1}1\right]dz=\frac{8ab^2\pi}{4b}\int_{z=-c}^{c} z\left(1-\frac{z^2}{c^2}\right)dz$$

$=0$ By the property of definite integral.

Q.20. If $\mathbf{F}=x\,\mathbf{i}-y\,\mathbf{j}+(z^2-1)\,\mathbf{k}$, find the value of $\iint_S \mathbf{F}\bullet\mathbf{n}\,dS$ where S is the closed surface bounded by the planes $z=0$, $z=1$ and the cylinder $x^2+y^2=4$.

Ans. By divergence theorem, we have $\iint_S \mathbf{F}\bullet\mathbf{n}\,dS=\iiint_V \text{div}\,\mathbf{F}\,dV$.

Here $\text{div}\,\mathbf{F}=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(-y)+\frac{\partial}{\partial z}(z^2-1)=1-1+2z=2z.$

$$\therefore\quad \iiint_V \text{div}\,\mathbf{F}\,dV=\int_{z=0}^{1}\int_{y=-2}^{2}\int_{x=-\sqrt{(4-y^2)}}^{\sqrt{(4-y^2)}} 2z\,dx\,dy\,dz$$

$$=\int_{z=0}^{1}\int_{y=-2}^{2}[2zx]_{x=-\sqrt{(4-y^2)}}^{\sqrt{(4-y^2)}}\,dy\,dz$$

$$=\int_{z=0}^{1}\int_{y=-2}^{2} 4z\sqrt{(4-y^2)}\,dy\,dz=\int_{y=-2}^{2}\left[4\frac{z^2}{2}\sqrt{(4-y^2)}\right]_{z=0}^{1} dy$$

$$=2\int_{y=-2}^{2}\sqrt{(4-y^2)}\,dy=4\int_{0}^{2}\sqrt{(4-y^2)}\,dy=4\left[\frac{y}{2}\sqrt{(4-y^2)}+2\sin^{-1}\frac{y}{2}\right]_0^2$$

$$=4\,[2\sin^{-1}1]=4\,(2)\frac{\pi}{2}=4\pi.$$

Q.21. (i) For any closed surfae S, prove that $\iint_S \text{curl}\,\mathbf{F}\bullet\mathbf{n}\,dS=0.$

(ii) Evaluate $\iint_S \mathbf{r}\bullet\mathbf{n}\,dS$, where S is a closed surface.

(iii) If $\mathbf{F}=ax\,\mathbf{i}+by\,\mathbf{j}+cz\,\mathbf{k}$, a, b, c are constants show that

$$\iint_S \mathbf{F}\bullet\mathbf{n}\,dS=\frac{4}{3}\pi(a+b+c),$$

where S is the surface of a unit sphere.

Ans. (i) By divergence theorem, we have

$$\iint_S \text{curl}\,\mathbf{F}\bullet\mathbf{n}\,dS=\iiint_V (\text{div curl}\,\mathbf{F})\,dV,\quad \text{where } V \text{ is the volume enclosed by } S$$

$$=0,\ \text{since div curl}\,\mathbf{F}=0.$$

(ii) By the divergence theorem, we have

$$\iint_S \mathbf{r}\bullet\mathbf{n}\,dS=\iiint_V \nabla\bullet\mathbf{r}\,dV=\iiint_V 3\,dV,\ \text{since } \nabla\bullet\mathbf{r}=\text{div }\mathbf{r}=3$$

$$=3V,\ \text{where } V \text{ is the volume enclosed by } S.$$

(iii) By the divergence theorem, we have

$$\iint_S \mathbf{F}\bullet\mathbf{n}\,dS=\iiint_V (\nabla\bullet\mathbf{F})\,dV,\ \text{where } V \text{ is the volume enclosed by } S$$

$$=\iiint_V [\nabla\bullet(a\,x\,\mathbf{i}+by\,\mathbf{j}+cz\,\mathbf{k})]\,dV=\iiint_V\left[\frac{\partial}{\partial x}(a\,x)+\frac{\partial}{\partial y}(by)+\frac{\partial}{\partial z}(cz)\right]dV$$

$$=\iiint_V (a+b+c)\,dV=(a+b+c)\,V=(a+b+c)\frac{4}{3}\pi,$$

since the volume V enclosed by a sphere of unit radius is equal to $\frac{4}{3}\pi(1)^3$ *i.e.,* $\frac{4}{3}\pi.$

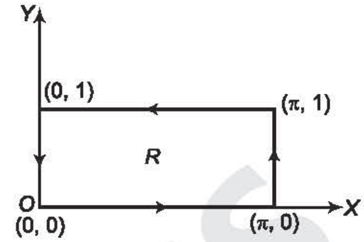
Q.22. Evaluate by Green's theorem $\oint_C (x^2-\cosh y)\,dx+(y+\sin x)\,dy$, where C is the rectangle with vertices $(0,0),(\pi,0),(\pi,1),(0,1)$.

Ans. By Green's theorem in plane, we have

$$\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy=\oint_C (M\,dx+N\,dy).$$

Here $M=x^2-\cosh y, N=y+\sin x.$

$\therefore \qquad \dfrac{\partial N}{\partial x} = \cos x, \dfrac{\partial M}{\partial y} = -\sinh y.$



Hence the given line integral is equal to

$$\iint_R (\cos x + \sinh y)\, dx\, dy$$

$$= \int_{x=0}^{\pi} \int_{y=0}^{1} (\cos x + \sinh y)\, dy\, dx$$

$$= \int_{x=0}^{\pi} \left[y \cos x + \cosh y \right]_{y=0}^{1} dx$$

$$= \int_{x=0}^{\pi} [\cos x + \cosh 1 - 1]\, dx = \left[\sin x + x \cosh 1 - x \right]_0^{\pi} = (\cosh 1 - 1).$$

SECTION-C LONG ANSWER TYPE QUESTIONS

Q.1. If r is a vector function of a scalar *t* and a is a constant vector, *m* a constant, differentiate the following with respect to *t* :

(i) $\mathbf{r} \bullet \mathbf{a}$, (ii) $\mathbf{r} \times \mathbf{a}$, (iii) $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt}$, (iv) $\mathbf{r} \bullet \dfrac{d\mathbf{r}}{dt}$,

(v) $\mathbf{r}^2 + \dfrac{1}{\mathbf{r}^2}$, (vi) $m\left(\dfrac{d\mathbf{r}}{dt}\right)^2$, (vii) $\dfrac{\mathbf{r} + \mathbf{a}}{\mathbf{r}^2 + \mathbf{a}^2}$.

Ans. (i) Let $R = \mathbf{r} \bullet \mathbf{a}$. [Note that $\mathbf{r} \bullet \mathbf{a}$ is a scalar]

Then $\dfrac{d\mathbf{R}}{dt} = \dfrac{d\mathbf{r}}{dt} \bullet \mathbf{a} + \mathbf{r} \bullet \dfrac{d\mathbf{a}}{dt} = \dfrac{d\mathbf{r}}{dt} \bullet \mathbf{a} + \mathbf{r} \bullet \mathbf{0}$ $\left[\because \dfrac{d\mathbf{a}}{dt} = \mathbf{0}, \text{as } \mathbf{a} \text{ is constant}\right]$

$$= \frac{d\mathbf{r}}{dt} \bullet \mathbf{a} + 0 = \frac{d\mathbf{r}}{dt} \bullet \mathbf{a}.$$

(ii) Let $\mathbf{R} = \mathbf{r} \times \mathbf{a}$.

Then $\dfrac{d\mathbf{R}}{dt} = \dfrac{d\mathbf{r}}{dt} \times \mathbf{a} + \mathbf{r} \times \dfrac{d\mathbf{a}}{dt} = \dfrac{d\mathbf{r}}{dt} \times \mathbf{a} + \mathbf{r} \times \mathbf{0}$ $\left[\because \dfrac{d\mathbf{a}}{dt} = \mathbf{0}\right]$

$$= \frac{d\mathbf{r}}{dt} \times \mathbf{a} + \mathbf{0} = \frac{d\mathbf{r}}{dt} \times \mathbf{a}.$$

(iii) Let $\mathbf{R} = \mathbf{r} \times \dfrac{d\mathbf{r}}{dt}$.

Then $\dfrac{d\mathbf{R}}{dt} = \dfrac{d\mathbf{r}}{dt} \times \dfrac{d\mathbf{r}}{dt} + \mathbf{r} \times \dfrac{d^2\mathbf{r}}{dt^2} = \mathbf{0} + \mathbf{r} \times \dfrac{d^2\mathbf{r}}{dt^2}$ $\left[\because \dfrac{d\mathbf{r}}{dt} \times \dfrac{d\mathbf{r}}{dt} = \mathbf{0}\right]$

$$= \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}.$$

(iv) Let $R = \mathbf{r} \bullet \dfrac{d\mathbf{r}}{dt}$.

Then $\dfrac{dR}{dt} = \dfrac{d\mathbf{r}}{dt} \bullet \dfrac{d\mathbf{r}}{dt} + \mathbf{r} \bullet \dfrac{d^2\mathbf{r}}{dt^2} = \left(\dfrac{d\mathbf{r}}{dt}\right)^2 + \mathbf{r} \bullet \dfrac{d^2\mathbf{r}}{dt^2}.$

(v) Let $R = \mathbf{r}^2 + \frac{1}{\mathbf{r}^2}$.

Then $\frac{dR}{dt} = \frac{d}{dt}(\mathbf{r}^2) + \frac{d}{dt}\left(\frac{1}{\mathbf{r}^2}\right) = \frac{d}{dt}(r^2) + \frac{d}{dt}\left(\frac{1}{r^2}\right)$, where $r = |\mathbf{r}|$

$$= 2r\frac{dr}{dt} - \frac{2}{r^3}\frac{dr}{dt}.$$

(vi) Let $R = m\left(\frac{d\mathbf{r}}{dt}\right)^2$

Then $\frac{dR}{dt} = m\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)^2 = 2m\frac{d\mathbf{r}}{dt} \bullet \frac{d^2\mathbf{r}}{dt^2}$ $\left[\text{Note}: \frac{dr^2}{dt} = 2r \bullet \frac{dr}{dt}\right]$

$$= 2m\frac{d\mathbf{r}}{dt} \bullet \frac{d^2\mathbf{r}}{dt^2}.$$

(vii) Let $\mathbf{R} = \frac{\mathbf{r} + \mathbf{a}}{\mathbf{r}^2 + \mathbf{a}^2}$.

Then $\frac{d\mathbf{R}}{dt} = \frac{1}{(\mathbf{r}^2 + \mathbf{a}^2)}\frac{d}{dt}(\mathbf{r} + \mathbf{a}) + \left\{\frac{d}{dt}\left(\frac{1}{\mathbf{r}^2 + \mathbf{a}^2}\right)\right\}(\mathbf{r} + \mathbf{a})$

[Note that $\mathbf{r}^2 + \mathbf{a}^2$ is a scalar]

$$= \frac{1}{\mathbf{r}^2 + \mathbf{a}^2}\left(\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{a}}{dt}\right) - \left\{\frac{1}{(\mathbf{r}^2 + \mathbf{a}^2)^2}\frac{d}{dt}(\mathbf{r}^2 + \mathbf{a}^2)\right\}(\mathbf{r} + \mathbf{a})$$

$$= \frac{1}{\mathbf{r}^2 + \mathbf{a}^2}\frac{d\mathbf{r}}{dt} - \frac{2\,\mathbf{r} \bullet \frac{d\mathbf{r}}{dt}}{(\mathbf{r}^2 + \mathbf{a}^2)^2}(\mathbf{r} + \mathbf{a}) \quad \left[\because \frac{d\mathbf{a}}{dt} = 0, \frac{d}{dt}\mathbf{r}^2 = 2\mathbf{r} \bullet \frac{d\mathbf{r}}{dt}, \frac{d}{dt}\mathbf{a}^2 = 0\right]$$

Q.2. If $r = |\boldsymbol{r}|$ where $\boldsymbol{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

(i) $\nabla f(r) = f'(r)\nabla r$ (ii) $\nabla r = \frac{\boldsymbol{r}}{r}$ (iii) $\nabla f(r) \times \boldsymbol{r} = 0$

(iv) $\nabla r^n = nr^{n-2}\boldsymbol{r}$ (v) $\nabla r^{-3} = -3r^{-5}\boldsymbol{r}$.

Ans. (i) Since we know that

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}. \quad ...(1)$$

$\therefore$ $$\nabla f(r) = \frac{\partial}{\partial x}(f(r))\hat{i} + \frac{\partial}{\partial y}(f(r))\hat{j} + \frac{\partial}{\partial z}(f(r))\hat{k}$$

or $$\nabla f(r) = f'(r)\frac{\partial r}{\partial x}\hat{i} + f'(r)\frac{\partial r}{\partial y}\hat{j} + f'(r)\frac{\partial r}{\partial z}\hat{k}$$

$$\nabla f(r) = f'(r)\left[\frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}\right]$$

$\nabla f(r) = f'(r)\nabla r$ [using (i)] **[Remember]**

(ii) $\nabla r = \frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}$ (by definition of gradient)

Since $\boldsymbol{r} = x\hat{i} + y\hat{j} + z\hat{k}.$

$\therefore$ $|\boldsymbol{r}|^2 = x^2 + y^2 + z^2$

or $r^2 = x^2 + y^2 + z^2$ $(\because |\boldsymbol{r}| = r)$

$\therefore$ $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$\therefore$ $\nabla r = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k} = \frac{1}{r}(x\hat{i} + y\hat{j} + z\hat{k})$ or $\nabla r = \frac{\boldsymbol{r}}{r}.$

(iii) $\nabla f(r) = f'(r)\nabla r$ [from (i)]

$= f'(r)\frac{\boldsymbol{r}}{r}$ [from (ii)]

Now $\nabla f(r) \times \boldsymbol{r} = \frac{f'(r)}{r}\boldsymbol{r} \times \boldsymbol{r} = \boldsymbol{0}$ $(\because \boldsymbol{r} \times \boldsymbol{r} = \boldsymbol{0})$

(iv) Since $\nabla f(r) = f'(r)\nabla r.$ [from (i)]

Let $f(r) = r^n.$

$\therefore$ $\nabla r^n = nr^{n-1}\nabla r = nr^{n-1}\left(\frac{\boldsymbol{r}}{r}\right)$ $\left(\because \nabla r = \frac{\boldsymbol{r}}{r}\right)$

$= nr^{n-2}\boldsymbol{r}$

or $\nabla r^n = nr^{n-2}\boldsymbol{r}.$

(v) From part (iv) $\nabla r^n = nr^{n-2}\boldsymbol{r} \Rightarrow \nabla r^{-3} = -3r^{-3-2}\boldsymbol{r} = -3r^{-5}\boldsymbol{r}.$

Q.3. Prove that $\text{curl}(\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{b}.\nabla)\boldsymbol{a} - \boldsymbol{b}\,\text{div}\,\boldsymbol{a} - (\boldsymbol{a}.\nabla)\boldsymbol{b} + \boldsymbol{a}\,\text{div}\,\boldsymbol{b}.$

Ans. We have $\text{curl}(\boldsymbol{a} \times \boldsymbol{b}) = \nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (\boldsymbol{a} \times \boldsymbol{b})$

$$= \hat{i} \times \frac{\partial}{\partial x}(\boldsymbol{a} \times \boldsymbol{b}) + \hat{j} \times \frac{\partial}{\partial y}(\boldsymbol{a} \times \boldsymbol{b}) + \hat{k} \times \frac{\partial}{\partial z}(\boldsymbol{a} \times \boldsymbol{b})$$

$$= \hat{i} \times \left(\frac{\partial \boldsymbol{a}}{\partial x} \times \boldsymbol{b} + \boldsymbol{a} \times \frac{\partial \boldsymbol{b}}{\partial x}\right) + \hat{j} \times \left(\frac{\partial \boldsymbol{a}}{\partial y} \times \boldsymbol{b} + \boldsymbol{a} \times \frac{\partial \boldsymbol{b}}{\partial y}\right) + \hat{k} \times \left(\frac{\partial \boldsymbol{a}}{\partial z} \times \boldsymbol{b} + \boldsymbol{a} \times \frac{\partial \boldsymbol{b}}{\partial z}\right)$$

$$= \hat{i} \times \left(\frac{\partial \boldsymbol{a}}{\partial x} \times \boldsymbol{b}\right) + \hat{i} \times \left(\boldsymbol{a} \times \frac{\partial \boldsymbol{b}}{\partial x}\right) + \hat{j} \times \left(\frac{\partial \boldsymbol{a}}{\partial y} \times \boldsymbol{b}\right) + \hat{j} \times \left(\boldsymbol{a} \times \frac{\partial \boldsymbol{b}}{\partial y}\right)$$

$$+ \hat{k} \times \left(\frac{\partial \boldsymbol{a}}{\partial z} \times \boldsymbol{b}\right) + \hat{k} \times \left(\boldsymbol{a} \times \frac{\partial \boldsymbol{b}}{\partial z}\right)$$

$$=\left[\hat{i}\times\left(\frac{\partial \boldsymbol{a}}{\partial x}\times \boldsymbol{b}\right)+\hat{j}\times\left(\frac{\partial \boldsymbol{a}}{\partial y}\times \boldsymbol{b}\right)+\hat{k}\times\left(\frac{\partial \boldsymbol{a}}{\partial z}\times \boldsymbol{b}\right)\right]$$

$$+\left[\hat{i}\times\left(\boldsymbol{a}\times\frac{\partial \boldsymbol{b}}{\partial x}\right)+\hat{j}\times\left(\boldsymbol{a}\times\frac{\partial \boldsymbol{b}}{\partial y}\right)+\hat{k}\times\left(\boldsymbol{a}\times\frac{\partial \boldsymbol{b}}{\partial z}\right)\right]$$

$$=\left[(\hat{i}\,.\,\boldsymbol{b})\frac{\partial \boldsymbol{a}}{\partial x}-\left(\hat{i}\,.\,\frac{\partial \boldsymbol{a}}{\partial x}\right)\boldsymbol{b}+(\hat{j}\,.\,\boldsymbol{b})\frac{\partial \boldsymbol{a}}{\partial y}-\left(\hat{j}\,.\,\frac{\partial \boldsymbol{a}}{\partial y}\right)\boldsymbol{b}+(\hat{k}\,.\,\boldsymbol{b})\frac{\partial \boldsymbol{a}}{\partial z}-\left(\hat{k}\,.\,\frac{\partial \boldsymbol{a}}{\partial z}\right)\boldsymbol{b}\right]$$

$$-\left[(\hat{i}\,.\,\boldsymbol{a})\frac{\partial \boldsymbol{b}}{\partial x}-\left(\hat{i}\,.\,\frac{\partial \boldsymbol{b}}{\partial x}\right)\boldsymbol{a}+(\hat{j}\,.\,\boldsymbol{a})\frac{\partial \boldsymbol{b}}{\partial y}-\left(\hat{j}\,.\,\frac{\partial \boldsymbol{b}}{\partial y}\right)\boldsymbol{a}+(\hat{k}\,.\,\boldsymbol{a})\frac{\partial \boldsymbol{b}}{\partial z}-\left(\hat{k}\,.\,\frac{\partial \boldsymbol{b}}{\partial z}\right)\boldsymbol{a}\right]$$

Using $\boldsymbol{a}\,.\,\boldsymbol{b}=\boldsymbol{b}\,.\,\boldsymbol{a}$, we get

$$=\left[\left(\boldsymbol{b}\,.\,\hat{i}\,\frac{\partial \boldsymbol{a}}{\partial x}+\boldsymbol{b}\,.\,\hat{j}\,\frac{\partial \boldsymbol{a}}{\partial y}+\boldsymbol{b}\,.\,\hat{k}\,\frac{\partial \boldsymbol{a}}{\partial z}\right)-\left(\hat{i}\,.\,\frac{\partial \boldsymbol{a}}{\partial x}+\hat{j}\,.\,\frac{\partial \boldsymbol{a}}{\partial y}+\hat{k}\,.\,\frac{\partial \boldsymbol{a}}{\partial z}\right)\boldsymbol{b}\right]$$

$$-\left[\left(\boldsymbol{a}\,.\,\hat{i}\,\frac{\partial \boldsymbol{b}}{\partial x}+\boldsymbol{a}\,.\,\hat{j}\,\frac{\partial \boldsymbol{b}}{\partial y}+\boldsymbol{a}\,.\,\hat{k}\,\frac{\partial \boldsymbol{b}}{\partial z}\right)-\left(\hat{i}\,.\,\frac{\partial \boldsymbol{b}}{\partial x}+\hat{j}\,.\,\frac{\partial \boldsymbol{b}}{\partial y}+\hat{k}\,.\,\frac{\partial \boldsymbol{b}}{\partial z}\right)\boldsymbol{a}\right]$$

$$=\left[\left(\boldsymbol{b}\,.\,\hat{i}\,\frac{\partial}{\partial x}+\boldsymbol{b}\,.\,\hat{j}\,\frac{\partial}{\partial y}+\boldsymbol{b}\,.\,\hat{k}\,\frac{\partial}{\partial z}\right)\boldsymbol{a}-(\nabla\,.\,\boldsymbol{a})\,\boldsymbol{b}\right]$$

$$-\left[\left(\boldsymbol{a}\,.\,\hat{i}\,\frac{\partial}{\partial x}+\boldsymbol{a}\,.\,\hat{j}\,\frac{\partial}{\partial y}+\boldsymbol{a}\,.\,\hat{k}\,\frac{\partial}{\partial z}\right)\boldsymbol{b}-(\nabla\,.\,\boldsymbol{b})\,\boldsymbol{a}\right]$$

$$=(\boldsymbol{b}\,.\,\nabla)\,\boldsymbol{a}-(\nabla\,.\,\boldsymbol{a})\,\boldsymbol{b}-(\boldsymbol{a}\,.\,\nabla)\,\boldsymbol{b}+(\nabla\,.\,\boldsymbol{b})\,\boldsymbol{a}$$

Hence, curl $(\boldsymbol{a}\times\boldsymbol{b})=(\boldsymbol{b}\,.\,\nabla)\,\boldsymbol{a}-\boldsymbol{b}\,\text{div}\,(\boldsymbol{a})-(\boldsymbol{a}\,.\,\nabla)\,\boldsymbol{b}+\boldsymbol{a}\,\text{div}\,(\boldsymbol{b})$. **[Remember]**

Q.4. If V is a differentiable vector function and f is a scalar point function, then
(i) $\text{div}(f\boldsymbol{V})=f\,\text{div}\,\boldsymbol{V}+\boldsymbol{V}\,.\,(\text{grad}\,f)$.
(ii) $\text{curl}(f\boldsymbol{V})=(\nabla f)\times\boldsymbol{V}+f(\nabla\times\boldsymbol{V})$.

Ans. (i) Since we know that

$$\text{div}\,(f\boldsymbol{V})=\nabla\,.\,(f\boldsymbol{V})=\left(\hat{i}\,\frac{\partial}{\partial x}+\hat{j}\,\frac{\partial}{\partial y}+\hat{k}\,\frac{\partial}{\partial z}\right).(f\boldsymbol{V})$$

$$=\hat{i}\,.\,\frac{\partial}{\partial x}(f\boldsymbol{V})+\hat{j}\,.\,\frac{\partial}{\partial y}(f\boldsymbol{V})+\hat{k}\,.\,\frac{\partial}{\partial z}(f\boldsymbol{V})$$

$$=\hat{i}\,.\left(\frac{\partial f}{\partial x}\boldsymbol{V}+f\,\frac{\partial \boldsymbol{V}}{\partial x}\right)+\hat{j}\,.\left(\frac{\partial f}{\partial y}\boldsymbol{V}+f\,\frac{\partial \boldsymbol{V}}{\partial y}\right)+\hat{k}\,.\left(\frac{\partial f}{\partial z}\boldsymbol{V}+f\,\frac{\partial \boldsymbol{V}}{\partial z}\right)$$

$$=\left[\hat{i}\left(\frac{\partial f}{\partial x}\boldsymbol{V}\right)+\hat{j}\,.\left(\frac{\partial f}{\partial y}\boldsymbol{V}\right)+\hat{k}\,.\left(\frac{\partial f}{\partial z}\boldsymbol{V}\right)\right]+\left[\hat{i}\,.\left(f\,\frac{\partial \boldsymbol{V}}{\partial x}\right)+\hat{j}\,.\left(f\,\frac{\partial \boldsymbol{V}}{\partial y}\right)+\hat{k}\,.\left(\frac{\partial \boldsymbol{V}}{\partial z}f\right)\right]$$

$$=\left[\left(\frac{\partial f}{\partial x}\hat{i}\right).\boldsymbol{V}+\left(\frac{\partial f}{\partial y}\hat{j}\right).\boldsymbol{V}+\left(\frac{\partial f}{\partial z}\hat{k}\right).\boldsymbol{V}\right]+\left[f\left(\hat{i}\,.\,\frac{\partial \boldsymbol{V}}{\partial x}\right)+f\left(\hat{j}\,.\,\frac{\partial \boldsymbol{V}}{\partial y}\right)+f\left(\hat{k}\,.\,\frac{\partial \boldsymbol{V}}{\partial z}\right)\right]$$

$$=\left[\left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right).\boldsymbol{V}\right]+\left[f\left(\hat{i}\,.\frac{\partial \boldsymbol{V}}{\partial x}+\hat{j}\,.\frac{\partial \boldsymbol{V}}{\partial y}+\hat{k}\,.\frac{\partial \boldsymbol{V}}{\partial z}\right)\right]$$

$$=(\nabla f).\boldsymbol{V}+f\,(\nabla.\boldsymbol{V}).$$

$\therefore$ $\text{div}\,(f\,\boldsymbol{V})=f\,(\text{div}\,\boldsymbol{V})+\boldsymbol{V}\,.(\text{grad}\,f)$ **[Remember]**

(ii) $\text{Curl}\,(f\,\boldsymbol{V})=\nabla\times(f\,\boldsymbol{V})=\left(\hat{i}\,\frac{\partial}{\partial x}+\hat{j}\,\frac{\partial}{\partial y}+\hat{k}\,\frac{\partial}{\partial z}\right)\times(f\,\boldsymbol{V})$

$$=\left(\hat{i}\,\frac{\partial}{\partial x}\right)\times(f\,\boldsymbol{V})+\left(\hat{j}\,\frac{\partial}{\partial y}\right)\times(f\,\boldsymbol{V})+\left(\hat{k}\,\frac{\partial}{\partial z}\right)\times(f\,\boldsymbol{V})$$

$$=\hat{i}\times\frac{\partial}{\partial x}(f\,\boldsymbol{V})+\hat{j}\times\frac{\partial}{\partial y}(f\,\boldsymbol{V})+\hat{k}\times\frac{\partial}{\partial z}(f\,\boldsymbol{V})$$

$$=\hat{i}\times\left(\frac{\partial f}{\partial x}\boldsymbol{V}+f\,\frac{\partial \boldsymbol{V}}{\partial x}\right)+\hat{j}\times\left(\frac{\partial f}{\partial y}\boldsymbol{V}+f\,\frac{\partial \boldsymbol{V}}{\partial y}\right)+\hat{k}\times\left(\frac{\partial f}{\partial z}\boldsymbol{V}+f\,\frac{\partial \boldsymbol{V}}{\partial z}\right)$$

$$=\left[\left(\frac{\partial f}{\partial x}\hat{i}\right)\times\boldsymbol{V}+\left(\frac{\partial f}{\partial y}\hat{j}\right)\times\boldsymbol{V}+\left(\frac{\partial f}{\partial z}\hat{k}\right)\times\boldsymbol{V}\right]+\left[f\left(\hat{i}\times\frac{\partial \boldsymbol{V}}{\partial x}\right)+f\left(\hat{j}\times\frac{\partial \boldsymbol{V}}{\partial y}\right)+f\left(\hat{k}\times\frac{\partial \boldsymbol{V}}{\partial z}\right)\right]$$

$$=\left(\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right)\times\boldsymbol{V}+f\left(\hat{i}\times\frac{\partial \boldsymbol{V}}{\partial x}+\hat{j}\times\frac{\partial \boldsymbol{V}}{\partial y}+\hat{k}\times\frac{\partial \boldsymbol{V}}{\partial z}\right)$$

$$=(\nabla f)\times\boldsymbol{V}+f\,(\nabla\times\boldsymbol{V}).$$

$\therefore$ $\text{Curl}\,(f\,\boldsymbol{V})=(\nabla\,f)\times\boldsymbol{V}+f\,(\nabla\times\boldsymbol{V}).$ **[Remember]**

Q.5. If $\boldsymbol{u}=y\,\hat{\boldsymbol{i}}+z\,\hat{\boldsymbol{j}}+x\,\boldsymbol{k}$, $\boldsymbol{v}=xy\,\hat{\boldsymbol{i}}+yz\,\hat{\boldsymbol{j}}+zx\,\hat{\boldsymbol{k}}$, find
(i) curl $(\boldsymbol{u}\times\boldsymbol{v})$ (ii) $\boldsymbol{u}\times$ curl $\boldsymbol{v}$
(iii) $\boldsymbol{v}\times$ curl $\boldsymbol{u}$ (iv) div $(\boldsymbol{u}\times\boldsymbol{v})$.

Ans. (i) Since $\boldsymbol{u}=y\,\hat{i}+z\,\hat{j}+x\,\hat{k}$, $\boldsymbol{v}=xy\,\hat{i}+yz\,\hat{j}+zx\,\hat{k}$.

$$\therefore\quad \boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ y & z & x\\ xy & yz & zx\end{vmatrix}=\hat{i}\,(z^2x-xyz)+\hat{j}\,(x^2y-xyz)+\hat{k}\,(y^2z-xyz)$$

$$\text{Then curl}\,(\boldsymbol{u}\times\boldsymbol{v})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ (z^2x-xyz) & (x^2y-xyz) & (y^2x-xyz)\end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}(y^2z-xyz)-\frac{\partial}{\partial z}(x^2y-xyz)\right]+\hat{j}\left[\frac{\partial}{\partial z}(z^2x-xyz)-\frac{\partial}{\partial x}(y^2z-xyz)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(x^2y-xyz)-\frac{\partial}{\partial y}(z^2x-xyz)\right]$$

$$=\hat{i}\,[(2yz-xz)-(-xy)]+\hat{j}\,[2zx-xy+yz]+\hat{k}\,[2xy-yz+xz]$$

$$=(2yz-xz+xy)\,\hat{i}+(2zx-xy+yz)+\hat{k}\,(2xy-yz+xz).$$

(ii) $\text{curl } \boldsymbol{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}(zx) - \frac{\partial}{\partial z}(yz)\right] + \hat{j}\left[\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(zx)\right] + \hat{k}\left[\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy)\right]$$

$$= -y\hat{i} - z\hat{j} - x\hat{k} = -(y\hat{i} + z\hat{j} + x\hat{k}) = -\boldsymbol{u}.$$

$\therefore \quad \boldsymbol{u} \times \text{curl } \boldsymbol{v} = \boldsymbol{u} \times -\boldsymbol{u} = -(\boldsymbol{u} \times \boldsymbol{u}) = 0.$

(iii) $\text{curl } \boldsymbol{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \hat{i}(-1) + \hat{j}(-1) + \hat{k}(-1) = -(\hat{i} + \hat{j} + \hat{k}).$

$$\therefore \quad \boldsymbol{v} \times \text{curl } \boldsymbol{u} = (xy\hat{i} + yz\hat{j} + zx\hat{k}) \times \{-(\hat{i} + \hat{j} + \hat{k})\}$$
$$= -[xy\hat{k} - xy\hat{j} - yz\hat{k} + yz\hat{i} + zx\hat{j} - zx\hat{i}]$$
$$= \hat{i}(zx - yz) + \hat{j}(xy - zx) + (yz - xy)\hat{k}.$$

(iv) From (i) $\boldsymbol{u} \times \boldsymbol{v} = (z^2x - xyz)\hat{i} + (x^2y - xyz)\hat{j} + (y^2z - xyz)\hat{k}.$

$$\therefore \quad \text{div}(\boldsymbol{u} \times \boldsymbol{v}) = \frac{\partial}{\partial x}(z^2x - xyz) + \frac{\partial}{\partial y}(x^2y - xyz) + \frac{\partial}{\partial z}(y^2z - xyz)$$
$$= z^2 - yz + x^2 - xz + y^2 - xy = x^2 + y^2 + z^2 - xy - yz - zx.$$

Q.6. If a is a constant vector, prove that curl $\dfrac{\mathbf{a} \times \mathbf{r}}{r^3} = -\dfrac{\mathbf{a}}{r^3} + \dfrac{3\mathbf{r}}{r^5}(\mathbf{a} \bullet \mathbf{r})$.

Ans. We have $\text{curl } \dfrac{\mathbf{a} \times \mathbf{r}}{r^3} = \nabla \times \left(\dfrac{\mathbf{a} \times \mathbf{r}}{r^3}\right) = \Sigma \left\{\mathbf{i} \times \dfrac{\partial}{\partial x}\left(\dfrac{\mathbf{a} \times \mathbf{r}}{r^3}\right)\right\}.$

Now $\dfrac{\partial}{\partial x}\left(\dfrac{\mathbf{a} \times \mathbf{r}}{r^3}\right) = -\dfrac{3}{r^4}\dfrac{\partial r}{\partial x}(\mathbf{a} \times \mathbf{r}) + \dfrac{1}{r^3}\left(\mathbf{a} \times \dfrac{\partial \mathbf{r}}{\partial x}\right) + \dfrac{1}{r^3}\left(\dfrac{\partial \mathbf{a}}{\partial x} \times \mathbf{r}\right)$ ...(1)

Now $\dfrac{\partial \mathbf{a}}{\partial x} = \mathbf{0}$ because **a** is a constant vector.

Also $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}.$

$\therefore \quad \dfrac{\partial \mathbf{r}}{\partial x} = \mathbf{i}$. Further $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}.$

$\therefore$ (1) becomes $\dfrac{\partial}{\partial x}\left(\dfrac{\mathbf{a} \times \mathbf{r}}{r^3}\right) = -\dfrac{3}{r^4}\dfrac{x}{r}(\mathbf{a} \times \mathbf{r}) + \dfrac{1}{r^3}(\mathbf{a} \times \mathbf{i}) = -\dfrac{3x}{r^5}(\mathbf{a} \times \mathbf{r}) + \dfrac{1}{r^3}(\mathbf{a} \times \mathbf{i}).$

$$\therefore \quad \mathbf{i} \times \frac{\partial}{\partial x}\left(\frac{\mathbf{a} \times \mathbf{r}}{r^3}\right) = -\frac{3x}{r^5}\mathbf{i} \times (\mathbf{a} \times \mathbf{r}) + \frac{1}{r^3}\mathbf{i} \times (\mathbf{a} \times \mathbf{i})$$
$$= -\frac{3x}{r^5}[(\mathbf{i} \bullet \mathbf{r})\mathbf{a} - (\mathbf{i} \bullet \mathbf{a})\mathbf{r}] + \frac{1}{r^3}[(\mathbf{i} \bullet \mathbf{i})\mathbf{a} - (\mathbf{i} \bullet \mathbf{a})\mathbf{i}]$$

$$= -\frac{3x}{r^5} x\mathbf{a} + \frac{3x}{r^5} a_1 \mathbf{r} + \frac{1}{r^3}\mathbf{a} - \frac{1}{r^3} a_1 \mathbf{i}$$

$$[\because \mathbf{i} \bullet \mathbf{r} = x \text{ and } \mathbf{i} \bullet \mathbf{a} = a_1 \text{ if } \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}]$$

$$= -\frac{3x^2}{r^5}\mathbf{a} + \frac{3}{r^5} a_1 x\mathbf{r} + \frac{1}{r^3}\mathbf{a} - \frac{1}{r^3} a_1 \mathbf{i}.$$

$$\therefore \quad \Sigma\left\{\mathbf{i} \times \frac{\partial}{\partial x}\left(\frac{\mathbf{a} \times \mathbf{r}}{r^3}\right)\right\} = \left\{-\frac{3}{r^5}\Sigma x^2\right\}\mathbf{a} + \left\{\frac{3}{r^5}\Sigma a_1 x\right\}\mathbf{r} + \frac{3}{r^3}\mathbf{a} - \frac{1}{r^3}\Sigma a_1 \mathbf{i}$$

$$= -\frac{3}{r^5} r^2 \mathbf{a} + \frac{3}{r^5}(\mathbf{r} \bullet \mathbf{a})\mathbf{r} + \frac{3}{r^3}\mathbf{a} - \frac{1}{r^3}\mathbf{a}$$

$$[\because \Sigma x^2 = r^2, \Sigma a_1 x = \mathbf{r} \bullet \mathbf{a}, \Sigma a_1 \mathbf{i} = \mathbf{a}]$$

$$= -\frac{\mathbf{a}}{r^3} + \frac{3}{r^5}(\mathbf{a} \bullet \mathbf{r})\mathbf{r}.$$

Q.7. Verify divergence theorem for $\mathbf{F} = (x^2 - yz)\mathbf{i} + (y^2 - zx)\mathbf{j} + (z^2 - xy)\mathbf{k}$ taken over the rectangular parallelopiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.

Ans. We have div $\mathbf{F} = \nabla \bullet \mathbf{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy) = 2x + 2y + 2z.$

$\therefore$ volume integer $= \iiint_V \nabla \bullet \mathbf{F}\, dV = \iiint_V 2(x + y + z)\, dV$

$$= 2\int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a}(x + y + z)\, dx\, dy\, dz$$

$$= 2\int_{z=0}^{c}\int_{y=0}^{b}\left[\frac{x^2}{2} + yx + zx\right]_{x=0}^{a} dy\, dz$$

$$= 2\int_{z=0}^{c}\int_{y=0}^{b}\left[\frac{a^2}{2} + ay + az\right] dy\, dz$$

$$= 2\int_{z=0}^{c}\left[\frac{a^2}{2} y + a\frac{y^2}{2} + azy\right]_{y=0}^{b} dz = 2\int_{z=0}^{c}\left[\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right] dz$$

$$= 2\left[\frac{a^2 b}{2} z + \frac{ab^2}{2} z + ab\frac{z^2}{2}\right]_0^c = [a^2 bc + ab^2 c + abc^2]$$

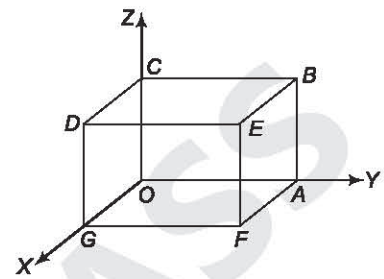
$$= abc\,(a + b + c).$$

Surface Integral : We shall now calculate $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$ over the six faces of the rectangular parallelopipied. Over the face $DEFG$, $\mathbf{n} = \mathbf{i}, x = a$.

Therefore, $\iint_{DEFG} \mathbf{F} \bullet \mathbf{n}\, dS$

$$=\int_{z=0}^{c}\int_{y=0}^{b}[(a^2-yz)\,\mathbf{i}+(y^2-za)\,\mathbf{j}+(z^2-ay)\,\mathbf{k}]\bullet\mathbf{i}\,dy\,dz$$

$$=\int_{z=0}^{c}\int_{y=0}^{b}(a^2-yz)\,dy\,dz$$



$$=\int_{z=0}^{c}\left[a^2y-z\frac{y^2}{2}\right]_{y=0}^{b}dz$$

$$=\int_{z=0}^{c}\left[a^2b-\frac{zb^2}{2}\right]dz$$

$$=\left[a^2bz-\frac{z^2}{4}b^2\right]_0^c=a^2bc-\frac{c^2b^2}{4}.$$

Over the face $ABCO$, $\mathbf{n}=-\mathbf{i}$, $x=0$. Therefore

$$\iint_{ABCO}\mathbf{F}\bullet\mathbf{n}\,dS=\iint[(0-yz)\,\mathbf{i}+\ldots+\ldots]\bullet(-\mathbf{i})\,dy\,dz$$

$$=\int_{z=0}^{c}\int_{y=0}^{b}y\,z\,dy\,dz=\int_{z=0}^{c}\left[\frac{y^2}{2}z\right]_{y=0}^{b}dz$$

$$=\int_{z=0}^{c}\frac{b^2}{2}z\,dz=\frac{b^2c^2}{4}.$$

Over the face $ABEF$, $\mathbf{n}=\mathbf{j}$, $y=b$. Therefore

$$\iint_{ABEF}\mathbf{F}\bullet\mathbf{n}\,dS=\int_{z=0}^{c}\int_{x=0}^{a}[(x^2-bz)\,\mathbf{i}+(b^2-zx)\,\mathbf{j}+(z^2-bx)\,\mathbf{k}]\bullet\mathbf{j}\,dx\,dz$$

$$=\int_{z=0}^{c}\int_{x=0}^{a}(b^2-zx)\,dx\,dz=b^2\,ca-\frac{a^2c^2}{4}.$$

Over the face $OGDC$, $\mathbf{n}=-\mathbf{j}$, $y=0$. Therefore

$$\iint_{OGDC}\mathbf{F}\bullet\mathbf{n}\,dS=\int_{z=0}^{c}\int_{x=0}^{a}zx\,dx\,dz=\frac{c^2a^2}{4}.$$

Over the face $BCDE$, $\mathbf{n}=\mathbf{k}$, $z=c$. Therefore

$$\iint_{BCDE}\mathbf{F}\bullet\mathbf{n}\,dS=\int_{y=0}^{b}\int_{x=0}^{a}(c^2-xy)\,dx\,dy=c^2ab-\frac{a^2b^2}{4}.$$

Over the face $AFGO$, $\mathbf{n}=-\mathbf{k}$, $z=0$. Therefore

$$\iint_{AFGO}\mathbf{F}\bullet\mathbf{n}\,dS=\int_{y=0}^{b}\int_{x=0}^{a}x\,y\,dx\,dy=\frac{a^2b^2}{4}.$$

Adding the six surface integrals, we get

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \left(a^2bc - \frac{c^2b^2}{4} + \frac{c^2b^2}{4}\right) + \left(b^2ca - \frac{a^2c^2}{4} + \frac{a^2c^2}{4}\right)$$

$$+ \left(c^2ab - \frac{a^2b^2}{4} + \frac{a^2b^2}{4}\right)$$

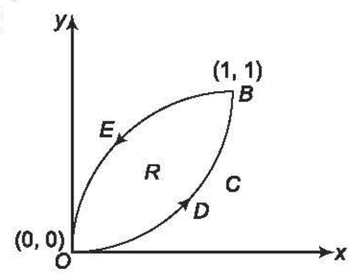
$$= abc\,(a + b + c).$$

Hence the theorem is verified.

Q.8. Verify Green's theorem in the plane for $\int_C [(2xy - x^2)dx + (x^2 + y^2)dy]$ where C is the boundary of the region enclosed by $y = x^2$ and $y^2 = x$ described in the positive sense.

Ans. Let R be the region enclosed by $y = x^2$ and $y^2 = x$ whose boundary C is traversed in the positive direction as shown in Fig.

The curves $y = x^2$ and $y^2 = x$ intersect at (0, 0) and (1, 1) and have



$$P(x, y) = 2xy - x^2$$

and $$Q(x, y) = x^2 + y^2.$$

$\therefore$ $$\frac{\partial P}{\partial y} = 2x \text{ and } \frac{\partial Q}{\partial x} = 2x.$$

By Green's theorem, we have

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int_C P\, dx + Q\, dy. \quad ...(1)$$

$$\therefore \text{ L.H.S.} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$= \iint_R (2x - 2x)\, dx\, dy = \iint_R 0 \,.\, dx\, dy = 0$$

and $$\text{R.H.S.} = \int_C (P\,dx + Q\,dy) = \int_C [(2xy - x^2)\,dx + (x^2 + y^2)\,dy]$$

$$= \int_{ODB} [(2xy - x^2)\,dx + (x^2 + y^2)\,dy] + \int_{BEO} [(2xy - x^2)\,dx + (x^2 + y^2)\,dy] \quad ...(2)$$

[$\because$ C consists of two curves ODB and BEO]

Along the curves BEO, we have

$y^2 = x$ and x varies from 0 to 1.

$\therefore$ $2y\,dy = dx.$

$\therefore$ $\int_{BEO} [(2xy - x^2)\,dx + (x^2 + y^2)\,dy]$

$$=\int_0^1 (2x^{3/2}-x^2)\,dx+\int_0^1 (x^2+x)\frac{dx}{2\sqrt{x}}$$

$$=\left[2\frac{x^{5/2}}{5/2}-\frac{x^3}{3}\right]_0^1+\frac{1}{2}\left[\frac{x^{5/2}}{5/2}+\frac{x^{3/2}}{3/2}\right]_0^1$$

$$=\left(\frac{4}{5}-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{2}{5}+\frac{2}{3}\right)=1$$

and along the curve *ODB*, we have

$y=x^2$ and x varies from 1 to 0.

$\therefore$ $dy=2x\,dx.$

$\therefore$ $\int_{ODB}[(2xy-x^2)\,dx+(x^2+y^2)\,dy)]$

$$=\int_1^0 [2x^3-x^2+2x^3+2x^5]\,dx$$

$$=\int_1^0 (4x^3-x^2+2x^5)\,dx=-\int_0^1 (4x^3-x^2+2x^5)\,dx$$

(by the property of definite integral)

$$=-\left[x^4-\frac{x^3}{3}+\frac{x^6}{3}\right]_0^1=-\left[1-\frac{1}{3}+\frac{1}{3}\right]=-1.$$

$\therefore$ R.H.S. $=-1+1=0.$

Thus L.H.S. = R.H.S.

Hence Green's theorem is verified. ❑

MODEL PAPER

Differential Calculus & Integral Calculus

B.Sc.-I (SEM-I) [M.M. : 75

Note : Attempt all the sections as per instructions.

Section-A : Very Short Answer Type Questions

Instruction : Attempt all **FIVE** questions. Each question carries **3 Marks**. Very Short Answer is required, not exceeding 75 words.

1. Define the term 'Bounded intervals'.

2. Define homogeneous function.

3. Show that the sequence $\left\langle \frac{1}{n} \right\rangle$ converges to 0.

4. Find $\int_1^2 x^3\, dx$, using fundamental theorem of integral calculus.

5. Define gradient of a scalar field.

Section-B : Short Answer Type Questions

Instruction : Attempt all **TWO** questions out of the following 3 questions. Each question carries **7.5 Marks**. Short Answer is required not exceeding 200 words.

6. Give examples to show that the union of an infinite collection of closed sets is not necessarily closed.

Or Show that the pedal equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}-\frac{r^2}{a^2b^2}.$$

7. Write and prove Cauchy's second theorem on limits.

Or Test the convergence of the series

$$\frac{1}{x}-\frac{1}{x+a}+\frac{1}{x+2a}\ldots, x>0, a>0.$$

8. State and prove Darboux theorem.

Or Find the perimeter of the cardioid $r=a(1-\cos\theta)$.

Section-C : Long Answer Type Questions

Instruction : Attempt all **THREE** questions out of the following 5 questions. Each question carries **15 Marks**. Answer is required in detail, between 500-800 words.

9. Let $I_n = \left] -\frac{1}{n}, 1+\frac{1}{n} \right[$ be an open interval for each $n \in N$. Prove that $\bigcap_{n=1}^{\infty} I_n$ is not a *nbd* of each of its points.

Or State and prove Taylor's theorem.

10. Find the envelope of the family of curves $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$, where the parameters a and b are connected by the relation $a^p+b^p=c^p$.

Or Trace the curve $y^2(a+x)=x^2(a-x)$.

11. Show by applying Cauchy's convergent criterion that the sequence $\langle s_n \rangle$ given by $s_n=1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}$ diverges.

Or Show that the sequence $< s_n >$ defined by $s_1=1, s_{n+1}=\frac{4+3s_n}{3+2s_n}, n \in N$ is convergent and find its limit.

12. Discuss the convergence of the given integral $\int_0^\infty x^{n-1}e^{-x}\,dx$, if $n>0$.

Or Evaluate the following integrals

(i) $\int_0^1 x^4(1-x)^2\,dx$ (ii) $\int_0^a y^4\sqrt{a^2-y^2}\,dy$

(iii) $\int_0^2 x(8-x^3)^{1/3}\,dx$ (iv) $\int_0^\infty \frac{x\,dx}{1+x^6}$.

13. State and prove Dirichlet's theorem for n variables.

Or Verify divergence theorem for $\mathbf{F}=(x^2-yz)\,\mathbf{i}+(y^2-zx)\,\mathbf{j}+(z^2-xy)\,\mathbf{k}$ taken over the rectangular parallelopiped $0\le x\le a, 0\le y\le b, 0\le z\le c$. ❑