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सर्वश्रेष्ठ परीक्षा पथ-प्रदर्शक

B.Sc. *First Year* (Certificate Course) Semester-I

PHYSICS

Mathematical Physics and Newtonian Mechanics

SYLLABUS

PART-A : BASIC MATHEMATICAL PHYSICS

Contribution of Indian Scientists : Contributions of Aryabhata, Vikram Sarabhai, C.V. Raman, S.N. Bose, M.N. Shaha, Subrahmanyam, Chandrasekhar.

UNIT-I **Vector Algebra :** Coordinate rotation, reflection and inversion for defining scalars, vectors, pseudo-scalars and pseudo-vectors (include physical examples). Component form in 2D and 3D. Geometrical and physical interpretation of addition, subtraction, dot product, wedge product, cross product and triple product of vectors. Position, separation and displacement vectors.

UNIT-II **Vector Calculus :** Geometrical and physical interpretation of vector differentiation, Gradient, Divergence and Curl and their significance. Vector integration, Line, Surface (flux) and Volume integrals of vector fields. Gradient theorem, Gauss-divergence theorem, Stoke-curl theorem, Green's theorem (statement only). Introduction to Dirac delta function.

UNIT-III **Coordinate Systems :** 2D & 3D Cartesian, Spherical and Cylindrical coordinate systems, basis vectors, transformation equations. Expressions for displacement vector, arc length, area element, volume element, gradient, divergence and curl in different coordinate systems. Components of velocity and acceleration in different coordinate systems.

UNIT-IV **Introduction to Tensors :** Principle of invariance of physical laws w.r.t. different coordinate systems as the basis for defining tensors. contravariant, covariant & mixed tensors and their ranks, 4-vectors. Index notation and summation convention. Symmetric and skew-symmetric tensors. Examples of tensors in physics.

PART-B : NEWTONIAN MECHANICS & WAVE MOTION

UNIT-V **Dynamics of a System of Particles :** Review of historical development of mechanics up to Newton. Background, statement and critical analysis of Newton's axioms of motion. Dynamics of a system of particles, centre of mass motion, and conservation laws & their deductions. Rotating frames of reference.

UNIT-VI **Dynamics of a Rigid Body :** Angular momentum, Torque, Rotational energy and the inertia tensor. Rotational inertia for simple bodies (ring, disk, rod, solid and hollow sphere, solid and hollow cylinder, rectangular lamina). The combined translational and rotational motion of a rigid body on horizontal and inclined planes. Elasticity, relations between elastic constants, bending of beam and torsion of cylinder.

UNIT-VII **Motion of Planets and Satellites :** Two particle central force problem, reduced mass, relative and centre of mass motion. Newton's law of gravitation, gravitational field and gravitational potential. Kepler's laws of planetary motion and their deductions. Motions of geo-synchronous & geo-stationary satellites and basic idea of Global Positioning System (GPS).

UNIT-VIII **Wave Motion :** Differential equation of simple harmonic motion and its solution, use of complex notation, damped and forced oscillations, Quality factor. Composition of simple harmonic motion, Lissajous figures. Differential equation of wave motion. Plane progressive waves in fluid media, reflection of waves and phase change, pressure and energy distribution. Principle of superposition of waves, stationary waves, phase and group velocity.

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CONTENTS

Part-A : Basic Mathematical Physics

UNIT-I

Vector Algebra

SECTION-A VERY SHORT ANSWER TYPE QUESTIONS

Q.1. Explain the meaning of displacement vector.

Ans. If any vector starts from $P=(x_1, y_1, z_1)$ at time $t=t_1$ and ends at $Q=(x_2, y_2, z_2)$ at time $t=t_2$ as shown in fig. 1., then the vector drawn from initial location to final location is called displacement vector in the given interval of time.

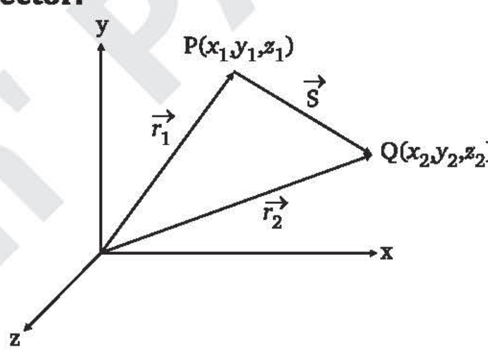


Fig. 1

Here, $\vec{PQ}=\vec{S}=$ displacement vector.

$\Rightarrow \quad \vec{S}=\vec{r_2}-\vec{r_1}$

where $\vec{r_1}=$ position vector of point P and

$\vec{r_2}=$ position vector of point Q.

Q.2. Position vector of a particle is $\vec{r}(t)=4t^2\hat{i}+5t\hat{j}-3\hat{k}$ meter, where t is measured in second. Calculate the displacement of particle in time interval $t=1s$ and $t=4s$.

Sol. Given position vector

$$\vec{r}(t)=4t^2\hat{i}+5t\hat{j}-3\hat{k} \text{ meter}$$

$\therefore$ position vector at $t=1s$

$$\vec{r_1}=4(1)^2\hat{i}+5(1)\hat{j}-3\hat{k} \text{ meter}$$

and position vector at $t=4s$

$$\vec{r_2}=4(4)^2\hat{i}+5(4)\hat{j}-3\hat{k} \text{ meter}$$
$$=64\hat{i}+20\hat{j}-3\hat{k} \text{ meter.}$$

Hence, displacement vector

$$\vec{S}=\vec{r_2}-\vec{r_1}=(64-4)\hat{i}+(20-5)\hat{j}+[(-3)-(-3)]\hat{k}$$
$$=\mathbf{60\hat{i}+15\hat{j}} \textbf{ meter.}$$

Q.3. Calculate the components of a unit vector that lies in the x-y plane and makes equal angle with positive direction of x and y-axis.

Sol. Let the given unit vector $\vec{A}$ lying in x-y plane, is

$$\hat{A} = A_x \hat{i} + A_y \hat{j}$$

If it makes equal angle α with x and y-axis

$$\alpha = 90° - \alpha \Rightarrow \alpha = 45°$$

$\therefore$ components of $\vec{A}$ are $\quad A_x = A_y = A\cos\alpha = (1)\cos 45° = \frac{1}{\sqrt{2}} \qquad (\because A = 1)$

Hence, we write $\quad \hat{A} = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$

Q.4. If two vectors $\vec{A}$ and $\vec{B}$ having equal magnitude of 10 units each and angle between then $60°$, then find :

(i) $\vec{A} + \vec{B}$ (ii) $\vec{A} - \vec{B}$

Sol. The given situation can be shown in the given figure.

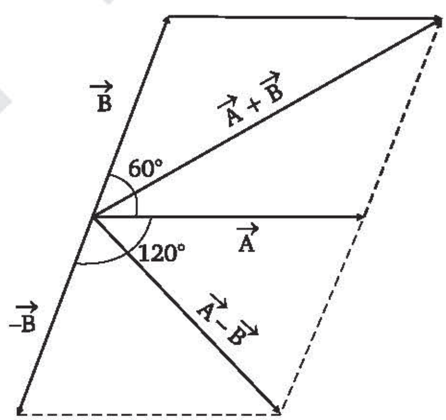


Fig.

(i) $$|\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$
$$= \sqrt{10^2 + 10^2 + 2 \times 10 \times 10 \times \cos 60°}$$
$$= \mathbf{10\sqrt{3}\ units}$$

(ii) $$|\vec{A} - \vec{B}| = \sqrt{10^2 + 10^2 - 2 \times 10 \times 10 \cos 120°}$$
$$= \mathbf{10\ units.}$$

Q.5. If $\vec{A} = (3, 5, -7)$ and $\vec{B} = (2, 7, 1)$, find $\vec{A} + \vec{B}$, $\vec{A} - \vec{B}$, $|\vec{A}|$, $|\vec{B}|$, $\vec{A} \cdot \vec{B}$ cosine of angle between $\vec{A}$ and $\vec{B}$.

Sol. Given $\vec{A} = (3, 5, -7) = 3\hat{i} + 5\hat{j} - 7\hat{k}$

and $\quad \vec{B} = (2, 7, 1) = 2\hat{i} + 7\hat{j} + \hat{k}$

(i) $$\vec{A} + \vec{B} = (3+2)\hat{i} + (5+7)\hat{j} + (-7+1)\hat{k}$$
$$= 5\hat{i} + 12\hat{j} - 6\hat{k} = \mathbf{(5, 12, -6)}$$

(ii) $$\vec{A} - \vec{B} = (3-2)\hat{i} + (5-7)\hat{j} + (-7-1)\hat{k}$$
$$= \hat{i} - 2\hat{j} - 8\hat{k} = \mathbf{(1, -2, -8)}$$

(iii) $|\vec{A}| = \sqrt{(3)^2 + (5)^2 + (-7)^2} = \sqrt{\mathbf{83}}$

(iv) $|\vec{B}| = \sqrt{(2)^2 + (7)^2 + (1)^2} = \sqrt{\mathbf{54}}$

(v) $\vec{A} \cdot \vec{B} = (3)(2) + (5)(7) + (-7)(1) = 6 + 35 - 7 = \mathbf{34}$

and $\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{AB} = \dfrac{34}{(\sqrt{83})(\sqrt{54})} \approx \mathbf{0.507}$

Q.6. If $\vec{A}, \vec{B}$ and $\vec{C}$ are three unit vectors such that $\vec{A} \times (\vec{B} \times \vec{C}) = \frac{1}{2}\vec{B}$, find the angle which $\vec{A}$ makes with $\vec{B}$ and $\vec{C}$, $\vec{B}$ and $\vec{C}$ being non-parallel.

Sol. Given : $\vec{A} \times (\vec{B} \times \vec{C}) = \frac{1}{2}\vec{B}$

By the vector triple product of vectors we have,

$$(\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} = \frac{1}{2}\vec{B}$$

The above equation holds only when

$$\vec{A} \cdot \vec{C} = \frac{1}{2} \text{ and } \vec{A} \cdot \vec{B} = 0$$

Angle between $\vec{A}$ and $\vec{C} = \cos^{-1}\left[\dfrac{\vec{A} \cdot \vec{C}}{AC}\right] = \cos^{-1}\left[\dfrac{1/2}{(1)(1)}\right] = \dfrac{\pi}{3}.$

$\vec{A} \cdot \vec{B} = 0$ implies that angle between $\vec{A}$ and $\vec{B} = \dfrac{\pi}{2}.$

Q.7. Let $\vec{A}$ be an arbitrary vector and $\hat{n}$ be a unit vector in some fixed direction. Show that $\vec{A} = (\vec{A} \cdot \hat{n})\hat{n} + (\hat{n} \times \vec{A}) \times \hat{n}$.

Sol.

$$\begin{aligned} \text{R.H.S.} &= (\vec{A} \cdot \hat{n})\hat{n} + (\hat{n} \times \vec{A}) \times \hat{n} \\ &= (\vec{A} \cdot \hat{n})\hat{n} + (\hat{n} \cdot \hat{n})\vec{A} - (\vec{A} \cdot \hat{n})\hat{n} \\ &= (\hat{n} \cdot \hat{n})\vec{A} = \mathbf{(1)}\vec{A} = \vec{A} = \textbf{L.H.S.} \end{aligned}$$

Q.8. Show that $(\vec{A} - \vec{B}) \cdot (\vec{A} + \vec{B}) = A^2 - B^2$.

Sol.

$$\begin{aligned} \text{L.H.S.} &= (\vec{A} - \vec{B}) \cdot (\vec{A} + \vec{B}) \\ &= \vec{A} \cdot \vec{A} + \vec{A} \cdot \vec{B} - \vec{B} \cdot \vec{A} - \vec{B}\,\vec{B} \end{aligned}$$

$= \vec{A} \cdot \vec{A} - \vec{B} \cdot \vec{B}$ $(\because \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{B})$

$= A^2 - B^2$ $(\because \vec{A} \cdot \vec{A} = A^2$ etc.)

= R.H.S.

SECTION-B SHORT ANSWER TYPE QUESTIONS

Q.1. Define zero vector or null vector. Mention its important properties.

Ans.

Zero Vector or Null Vector

A vector quantity which has the same initial and terminal is called zero or null vector. It is a vector of zero magnitude and since its length is zero (*i.e.*, its tip and tail coincide), **its direction can not be specified.**

Explanation : We know that, if we add two vectors $\vec{A}$ and $\vec{B}$, the result is a vector, not a scalar

i.e., $$\vec{A} + \vec{B} = \vec{C} \quad ...(1)$$

What happens when $\vec{B} = -\vec{A}$? The result $(\vec{C})$ must be a vector of a zero magnitude. For mathematical consistency of equation (1) it is convenient to have a vector of zero magnitude.

A zero vector is represented as $\vec{0}$. Thus, we can write

$$\vec{A} - \vec{A} = \vec{0}$$

Properties of Null Vector

A zero vector has following main properties :

1. $|\vec{0}| = 0$, thus if $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ is a zero vector, $A_x = A_y = A_z = 0$ *i.e.*, **all the components of a zero vector are zero.**
2. For any vector $\vec{A}$

   $$\vec{A} + \vec{0} = \vec{A} \quad \text{[Addition]}$$

   $$\vec{A} \times \vec{0} = \vec{0} \quad \text{[Cross product]}$$
3. For any number λ

   $$\lambda \vec{0} = \vec{0}$$
4. Vector product of a vector by itself is a zero vector.

   $$\vec{A} \times \vec{A} = \vec{0}$$

Some Physical Examples of Zero Vector

1. For a body, kept at rest :

   If, Resultant force acting on the body *i.e.*, $\vec{F} = \vec{0}$

then, velocity of the body, $\vec{v} = \vec{0}$

or, Acceleration of the body, $\vec{a} = \vec{0}$

Also, linear momentum of the body, $\vec{p} = m\vec{v} = \vec{0}$.

2. When a body is thrown vertically upward then its final velocity at highest point, $\vec{v} = \vec{0}$.

Q.2. Define a unit vector. Give unit vectors used in rectangular coordinate system.

Ans.

Unit Vector

A unit vector is a vector **whose magnitude is exactly 1 and points in a particular direction.** So, we can say a unit vector has a unit length and used to give direction of a vector. It is represented by putting '^' (cap) on the vector. So, if A is a unit vector then it is represented by '$\hat{A}$' & can read as "***A* cap**".

If $\vec{A}$ is a given vector, unit vector in the direction of $\vec{A}$ is represented as $\hat{A}$ and is given as :

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A}$$

or we may write, $\vec{A} = A\hat{A} = |\vec{A}|\hat{A}$

Points Regarding Unit Vector :

1. A unit vector has no unit and dimension. It is used to specify the direction of a vector.
2. By definition of unit vector, $|\vec{A}| = 1$.

Unit vectors used in rectangular coordinate system : Three unit vectors in x-y-z coordinate system are indicated by $\hat{i}$, $\hat{j}$ and $\hat{k}$.

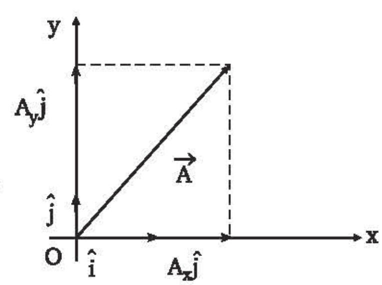


Fig. : Vector Decomposition into Component Vectors and Unit Vectors

They represent unit vectors pointing towards the positive direction of the corresponding axis. Decomposition of a vector in x-y plane into its component vectors and its unit vectors are illustrated by Figure.

The relation between unit vectors and component of vector $\vec{A}$ in the coordinate systems are given as :

$$\vec{A_x} = A_x\hat{i} \quad ...(1)$$

$$\vec{A_y} = A_y\hat{j} \quad ...(2)$$

$$\vec{A_z} = A_z\hat{k} \quad ...(3)$$

in equation (1), (2) and (3) $\hat{i}$, $\hat{j}$ and $\hat{k}$ represents the direction of $\vec{A_x}$, $\vec{A_y}$ and $\vec{A_z}$ respectively.

Q.3. Given two vectors $\vec{A} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{B} = -\hat{i} + 2\hat{j} + 6\hat{k}$. Calculate :

(a) Magnitudes of $\vec{A}$ and $\vec{B}$

(b) $(\vec{A} + \vec{B})$ and $(\vec{A} - \vec{B})$

(c) $\vec{A} \cdot \vec{B}$

(d) Angle between $\vec{A}$ and $\vec{B}$

(e) $\vec{A} \times \vec{B}$

(f) Projection of $\vec{A}$ on $\vec{B}$

(g) Angle between $\vec{B}$ and y-axis.

Sol. Given $\vec{A} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{B} = -\hat{i} + 2\hat{j} + 6\hat{k}$

Magnitudes of $\vec{A}$ and $\vec{B}$

(a) $$|\vec{A}| = A = \sqrt{(3)^2 + (4)^2 + (-5)^2} = \mathbf{\sqrt{50}}$$

and $$|\vec{B}| = B = \sqrt{(-1)^2 + (2)^2 + (6)^2} = \mathbf{\sqrt{41}}$$

(b) $$\vec{A} + \vec{B} = [(3) + (-1)]\hat{i} + [(4) + (2)]\hat{j} + [(-5) + (6)]\hat{k} = \mathbf{2\hat{i} + 6\hat{j} + \hat{k}}$$

and $$\vec{A} - \vec{B} = [(3) - (-1)]\hat{i} + [(4) - (2)]\hat{j} + [(-5) - (-6)]\hat{k} = \mathbf{4\hat{i} + 2\hat{j} - 11\hat{k}}$$

(c) $$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

$$= (3)(-1) + (4)(2) + (-5)(6) = -3 + 8 - 30 = \mathbf{-25}$$

(d) Angle θ between $\vec{A}$ and $\vec{B}$ is given by

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{-25}{(\sqrt{50})(\sqrt{41})} = \frac{-25}{5\sqrt{82}} = \frac{-5}{\sqrt{82}}$$

$$\theta = \mathbf{\cos^{-1}\left[\frac{-5}{\sqrt{82}}\right]}$$

(e) $$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & -5 \\ -1 & 2 & 6 \end{vmatrix}$$

$$= \hat{i}(24 + 10) - \hat{j}(18 - 5) + \hat{k}(6 + 4) = \mathbf{34\hat{i} - 13\hat{j} + 10\hat{k}}$$

(f) Projection of $\vec{A}$ on $\vec{B}$

$$A_B = A\cos\theta = \frac{\vec{A} \cdot \vec{B}}{B} = \mathbf{\frac{-25}{\sqrt{41}}}$$

(g) Angle between $\vec{B}$ and y-axis

$$\theta_y = \cos^{-1}\left[\frac{B_y}{B}\right] = \cos^{-1}\left[\frac{2}{\sqrt{41}}\right]$$

Q.4. Prove that

(a) Area of a parallelogram with touching sides $\vec{A}$ and $\vec{B}$ is $|\vec{A}\times\vec{B}|$

(b) Area of a triangle with sides $\vec{A}$ and $\vec{B}$ is $\frac{1}{2}|\vec{A}\times\vec{B}|$

Sol. In the given figure, consider parallelogram $OABC$ with touching sides OA and OB represented in vector form as :

$\vec{OA} = \vec{A}, \quad \vec{OB} = \vec{B}, \quad |\vec{OA}| = A, \quad |\vec{OB}| = B$

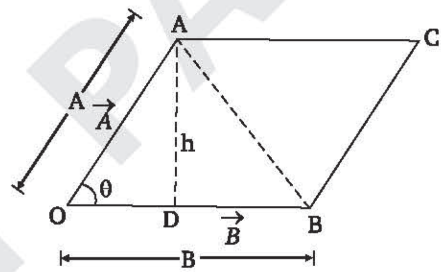


Fig.

(a) By geometry, area of parallelogram

$$= OB \times AD$$
$$= OB \times OA \sin\theta$$
$$= (B)(A)\sin\theta = AB\sin\theta$$
$$= |\vec{A}\times\vec{B}| \quad \textbf{Proved}$$

(b) In the given figure, consider the triangle OAB with sides OA and OB represented by vectors $\vec{A}$ and $\vec{B}$. By geometry, area of triangle

$$= \frac{1}{2} OB \times AD = \frac{1}{2}(B)(A\sin\theta)$$
$$= \frac{1}{2} AB\sin\theta = \frac{1}{2}|\vec{A}\times\vec{B}|$$

Proved.

Q.5. For three vectors $\vec{A}, \vec{B}$ and $\vec{C}$. Prove that $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$.

Sol. Let

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$
$$\vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$$

$$\vec{B} + \vec{C} = (B_x + C_x)\hat{i} + (B_y + C_y)\hat{j} + (B_z + C_z)\hat{k}$$

$$\text{L.H.S.} = \vec{A}\times(\vec{B}+\vec{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x + C_x & B_y + C_y & B_z + C_z \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \vec{A} \times \vec{B} + \vec{A} \times \vec{C} = \text{R.H.S.}$$

Hence Proved.

Thus, vector product is distributive with respect to addition.

Q.6. Deduce the necessary condition for two vectors $\vec{A} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{B} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ to be collinear.

Sol. Given, $\vec{A} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

and $\vec{B} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

Obviously, to be collinear (or parallel), the direction cosines of vector $\vec{A}$ must be equal to the respective direction cosines of vector $\vec{B}$.

i.e., we whould have $\frac{a_1}{A} = \frac{a_2}{B}, \frac{b_1}{A} = \frac{b_2}{B}, \frac{c_1}{A} = \frac{c_2}{B}$, which gives $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{A}{B}$.

Thus, the condition for collinearity of the two vectors works out to be that the ratio between the scalars a_1 and a_2, b_1 and b_2 and c_1 and c_2 must be the same, *i.e.*, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Alternatively, we could obtain the same condition by taking the cross product of two vectors and equating it to zero. Thus,

$$\vec{A} \times \vec{B} = (b_1c_2 - c_1b_2)\hat{i} + (c_1a_2 - a_1c_2)\hat{j} + (a_1b_2 - b_1a_2)\hat{k} = 0$$

which gives $b_1c_2 - c_1b_2 = 0, c_1a_2 - a_1c_2 = 0$ and $a_1b_2 - b_1a_2 = 0$, whence, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, as before.

Q.7. Show that :

(a) $(\vec{A} - \vec{B}) \cdot (\vec{A} + \vec{B}) = A^2 - B^2$ **(b) $(\vec{A} - \vec{B}) \times (\vec{A} + \vec{B}) = 2(\vec{A} \times \vec{B})$**

(c) $(\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B}) = A^2B^2 - (\vec{A} \cdot \vec{B})^2$

(d) $|\vec{A} \times \vec{B}|^2 + (\vec{A} \cdot \vec{B})^2 = A^2B^2$

Sol. (a) $\text{L.H.S.} = (\vec{A} - \vec{B}) \cdot (\vec{A} + \vec{B})$

$$= \vec{A} \cdot \vec{A} + \vec{A} \cdot \vec{B} - \vec{B} \cdot \vec{A} - \vec{B} \cdot \vec{B}$$

$$= \vec{A} \cdot \vec{A} - \vec{B} \cdot \vec{B} \qquad (\because \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A})$$

$= A^2 - B^2$ $\quad (\because \vec{A} \cdot \vec{A} = A^2 \text{ etc.})$

$=$ R.H.S.

(b) L.H.S. $= (\vec{A} - \vec{B}) \times (\vec{A} + \vec{B})$

$= (\vec{A} \times \vec{A}) + (\vec{A} \times \vec{B}) - (\vec{B} \times \vec{A}) - (\vec{B} \times \vec{B})$

$= (\vec{A} \times \vec{B}) - (\vec{B} \times \vec{A})$ $\quad [\because \vec{A} \times \vec{A} = \vec{0} \text{ etc.}]$

$= (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{B})$ $\quad [\because (\vec{B} \times \vec{A}) = -(\vec{A} \times \vec{B})]$

$= 2(\vec{A} \times \vec{B}) =$ R.H.S.

(c) L.H.S. $= (\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B})$

$= |\vec{A} \times \vec{B}|^2$ $\quad [\because \vec{A} \cdot \vec{A} = A^2]$

$= p(AB \sin\theta)^2 = A^2 B^2 \sin^2\theta$

$= A^2 B^2 (1 - \cos^2\theta) = A^2 B^2 - (AB\cos\theta)^2$

$= A^2 B^2 - (\vec{A} \cdot \vec{B})^2 =$ R.H.S.

(d) L.H.S. $= |\vec{A} \times \vec{B}|^2 + (\vec{A} \cdot \vec{B})^2$

$= (AB\sin\theta)^2 + (AB\cos\theta)^2 = A^2 B^2 (\sin^2\theta + \cos^2\theta)$

$= A^2 B^2 \cdot 1 = A^2 B^2 =$ R.H.S.

Q.8. If $\vec{A} = 2\hat{i} - \hat{j} - \hat{k}, \vec{B} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{C} = \hat{j} + \hat{k}$, find $(\vec{A} \cdot \vec{B})\vec{C}$, $\vec{A}(\vec{B} \cdot \vec{C})$, $\vec{A} \cdot (\vec{B} \times \vec{C}), (\vec{A} \times \vec{B}) \times \vec{C}$ and $\vec{A} \times (\vec{B} \times \vec{C})$.

Sol. Given, $\vec{A} = 2\hat{i} - \hat{j} - \hat{k}$

$\vec{B} = 2\hat{i} - 3\hat{j} + \hat{k}$

and $\vec{C} = \hat{j} + \hat{k}$

So, here, $\vec{A} \cdot \vec{B} = (2\hat{i} - \hat{j} - \hat{k}) \cdot (2\hat{i} - 3\hat{j} + \hat{k})$

$= (2)(2) + (-1)(-3) + (-1)(1) = 4 + 3 - 1 = 6$

and $\vec{A} \cdot \vec{C} = (2\hat{i} - \hat{j} - \hat{k}) \cdot (2\hat{i} - 3\hat{j} + \hat{k})$

$= (2)(0) + (-1)(1) + (-1)(1)$

$= 0 - 1 - 1 = -2$

also, $\vec{B}\cdot\vec{C}=(2\hat{i}-3\hat{j}+\hat{k})\cdot(\hat{j}+\hat{k})$

$$=(2)(0)+(-3)(1)+(1)(1)$$

$$=0-3+1=-2$$

$\therefore$ (i) $(\vec{A}\cdot\vec{B})\vec{C}=6(\hat{j}+\hat{k})=6\hat{j}+6\hat{k}$

(ii) $\vec{A}(\vec{B}\cdot\vec{C})=(2\hat{i}-\hat{j}-\hat{k})(-2)=-4\hat{i}+2\hat{j}+2\hat{k}$

(iii) $$\vec{A}\cdot(\vec{B}\times\vec{C})=\begin{vmatrix}2 & -1 & -1\\ 2 & -3 & 1\\ 0 & 1 & 1\end{vmatrix}$$

$$=2(-3-1)+1(2-0)-1(2-0)$$

$$=-8+2-2=-8$$

(iv) $(\vec{A}\times\vec{B})\times\vec{C}=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{B}\cdot\vec{C})\vec{A}$

$$=(-2)(2\hat{i}-3\hat{j}+\hat{k})-(-2)(2\hat{i}-\hat{j}-\hat{k})$$

$$=-4\hat{i}+6\hat{j}-2\hat{k}+4\hat{i}-2\hat{j}-2\hat{k}$$

$$=4\hat{j}-4\hat{k}$$

(v) $\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C}$

$$=(-2)(2\hat{i}-3\hat{j}+\hat{k})-(6)(\hat{j}+\hat{k})$$

$$=-4\hat{i}+6\hat{j}-2\hat{k}-6\hat{j}-6\hat{k}$$

$$=\mathbf{-4\hat{i}-8\hat{k}}$$

Q.9. Prove that if the sum of three vectors is zero, then their scalar triple product is zero.

Sol. Let $\vec{A}, \vec{B}$ and $\vec{C}$ are three vectors such that

$$\vec{A}+\vec{B}+\vec{C}=0$$

Hence $\vec{A}=-\vec{B}-\vec{C}$

$\Rightarrow$ $A_x=-B_x-C_x, A_y=-B_y-C_y$ and $A_z=-B_z-C_z$

$\therefore$ $$\vec{A}\cdot(\vec{B}\times\vec{C})=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

Substituting the values in determinant, we have

$$=\begin{vmatrix} -B_x - C_x & -B_y - C_y & -B_z - C_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$=-\begin{vmatrix} B_x & B_y & B_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} - \begin{vmatrix} C_x & C_y & C_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

∵ If two rows of a determinant are identical, its value is zero.

$$=0-0=0$$

Hence Proved.

Q.10. For four vectors $\vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$, prove that :

(i) $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})=(\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$

(ii) $(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D})=(\vec{A}\cdot\vec{C}\times\vec{D})\vec{B}-(\vec{B}\cdot\vec{C}\times\vec{D})\vec{A}$

$=(\vec{A}\cdot\vec{B}\times\vec{D})\vec{C}-(\vec{A}\cdot\vec{B}\times\vec{C})\vec{D}$

Sol. **(i)** Let $\vec{P}=\vec{A}\times\vec{B}$, then

$$\text{L.H.S.}=(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})=\vec{P}\cdot(\vec{C}\times\vec{D})$$

$$=(\vec{P}\times\vec{C})\cdot\vec{D}$$

(By interchanging dot and cross in scalar triple produces)

Substituting $\vec{P}$ we get,

$$\text{L.H.S.}=[(\vec{A}\times\vec{B})\times\vec{C}]\cdot\vec{D}=[(\vec{A}\cdot\vec{C})\vec{B}-(\vec{B}\cdot\vec{C})\vec{A}]\cdot\vec{D}$$

$$=(\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})=\text{R.H.S.}$$

(ii) Now again consider $\vec{P}=\vec{A}\times\vec{B}$

$$\text{L.H.S.}=(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D})$$

$$=\vec{P}\times(\vec{C}\times\vec{D})=(\vec{P}\cdot\vec{D})\cdot\vec{C}-(\vec{P}\cdot\vec{C})\vec{D}$$

Substituting $\vec{P}$ we get $\quad =(\vec{A}\times\vec{B}\cdot\vec{D})\vec{C}-(\vec{A}\times\vec{B}\cdot\vec{C})\vec{D}$

(by interchanging dot and cross in scalar product)

$$=(\vec{A}\cdot\vec{B}\times\vec{D})\vec{C}-(\vec{A}\cdot\vec{B}\times\vec{C})\vec{D} \qquad ...(1)$$

Again, let $\vec{Q} = \vec{C} \times \vec{D}$

Then, L.H.S. $= (\vec{A} \times \vec{B}) \times \vec{Q} = (\vec{A} \cdot \vec{Q})\vec{B} - (\vec{B} \cdot \vec{Q})\vec{A}$

Substituting $\vec{Q}$ we get $= (\vec{A} \cdot \vec{C} \times \vec{D})\vec{B} - (\vec{B} \cdot \vec{C} \times \vec{D})\vec{A}$ ...(2)

From equation (1) and (2)

$$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{B} \times \vec{D})\vec{C} - (\vec{A} \cdot \vec{B} \times \vec{C})\vec{D}$$

$$= (\vec{A} \times \vec{C} \times \vec{D})\vec{B} - (\vec{B} \cdot \vec{C} \times \vec{D})\vec{A}$$

Hence Proved.

Q.11. Define a pseudo vector. Show that cross product of two polar vector is pseudo vector while the vector triple product of three polar vector is a true vector.

Ans.

Pseudo Vector

A pseudo vector is a quantity that behaves like a vector under a proper rotation, but gains an additional sign change under an improper rotation (like reflection).

Geometrically, the direction of a reflected pseudo vector is opposite to its mirror image, but with equal magnitude. This is in contrast to a true or a polar vector which when reflected will match its mirror image.

A pseudovector is also refered as axial vector.

Angular velocity is one of the general example of a pseudo vector. The angular velocity, generally written as a vector has amplitude and a direction, however, on reflection or inversion, it acts differently than linear velocity, which is a true vector.

The most common examples of pseudo vectors are angular momentum, tourque, angular acceleration, magnetic field, magnetic dipole moment etc.

Cross product of two polar vectors is a pseudo vector

Consider two polar vectors $\vec{A}$ and $\vec{B}$. Under parity transformation of vector $\vec{A}$ and vector $\vec{B}$:

$$\vec{A} \xrightarrow{\text{P.T.}} -\vec{A} \text{ and } \quad \vec{B} \xrightarrow{\text{P.T.}} -\vec{B}$$

We know that

$(\vec{A} \times \vec{B})$ under parity transformation gives,

$$(\vec{A} \times \vec{B}) \xrightarrow{\text{P.T.}} (-\vec{A}) \times (-\vec{B}) = (\vec{A} \times \vec{B})$$

Since, $(\vec{A} \times \vec{B})$ does not changes sign under parity transformation (P.T.), hence the cross product of two polar vectors $\vec{A}$ & $\vec{B}$ (*i.e.*, $\vec{A} \times \vec{B}$) is a pseudo vector.

Vector triple product of three polar vectors is a true vector

Consider three polar vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. Under parity transformation of vector $\vec{A}$, $\vec{B}$ and $\vec{C}$:

$$\vec{A} \xrightarrow{\text{P.T.}} -\vec{A}$$

$$\vec{B} \xrightarrow{\text{P.T.}} -\vec{B}$$

and $$\vec{C} \xrightarrow{\text{P.T.}} -\vec{C}$$

$\therefore$ $$\vec{A} \times (\vec{B} \times \vec{C}) \xrightarrow{\text{P.T.}} -\vec{A} \times (-\vec{B} \times -\vec{C})$$

$$\longrightarrow -\vec{A} \times (\vec{B} \times \vec{C})$$

As under parity transformation $\vec{A} \times (\vec{B} \times \vec{C})$ reverses it sign, hence the vector triple product of the three polar vectors is a true vector.

Q.12. The position vectors of two particles ejected simultaneously from the same source are $\vec{r_1} = 3\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{r_2} = 2\hat{i} + 6\hat{j} + 8\hat{k}$. Obtain (i) the displacement $\vec{r}$ or particle two with respect to particle one, (ii) the magnitudes of $\vec{r_1}, \vec{r_2}$ and $\vec{r}$, (iii) angles between $\vec{r_1}$ and r_2, $\vec{r_1}$ and $\vec{r}$ and $\vec{r_2}$ and $\vec{r}$ and (iv) projection of $\vec{r}$ on $\vec{r_1}$.

Sol. (i) Here, obviously, displacement of particles 2 with respect to particle 1,

i.e., $$\vec{r} = \vec{r_2} - \vec{r_1} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}.$$

(ii) Magnitude of $\vec{r_1} = |\vec{r_1}| = \sqrt{9+16+25} = 5\sqrt{2} = 7.07,$

Magnitude of $\vec{r_2} = |\vec{r_2}| = \sqrt{4+36+64} = \sqrt{104} = 10.20$

and Magnitude of $\vec{r} = |\vec{r}| = \sqrt{1+4+9} = \sqrt{14} = \mathbf{3.74}$

(iii) We have $$\vec{r_1} \cdot \vec{r_2} = |\vec{r_1}||\vec{r_2}| \cos\theta_{r_1 r_2}$$

hence $$\cos\theta_{r_1 r_2} = \frac{\vec{r_1} \cdot \vec{r_2}}{|\vec{r_1}||\vec{r_2}|}$$

$$= \frac{6+24+40}{(7.07)(10.20)} = 0.9706 \quad \therefore \quad \theta_{r_1 r} = 14.0°$$

Similarly, $$\cos\theta_{r_1 r} = \frac{\vec{r_1} \cdot \vec{r}}{|\vec{r_1}||\vec{r}|} = \frac{-3+8+15}{(7.07)(3.74)} = 0.7563 \quad \therefore \quad \theta_{r_1 r} = 49.9°$$

and $$\cos\theta_{r_2 r} = \frac{\vec{r_2} \cdot \vec{r}}{|\vec{r_2}||\vec{r}|} = \frac{-2+12+24}{(10.20)(3.74)} = 0.8913 \text{ and } \therefore \theta_{r_2 r} = 27.0°$$

(iv) And, projection of $\vec{r}$ on $\vec{r_1} = \vec{r} \cdot \vec{r_1}$ (where $\vec{r_1}$ is the unit vector in the direction of $\vec{r_1}$)

$$= \frac{\vec{r} \cdot \vec{r_1}}{|\vec{r_1}|} = \frac{20}{7.07} = 2.83$$

SECTION-C LONG ANSWER TYPE QUESTIONS

Q.1. Define the components of a vector. How does a vector represented in terms of Cartesian components?

Ans.

Components of a Vector

The projection of a vector along a given line is known as its component along that give line.

Consider a vector $\vec{OA} = \vec{A}$, having magnitude A. To find its projection on a given line PQ. Draw a perpendicular OO' and AA' on line PQ as shown in fig.1.

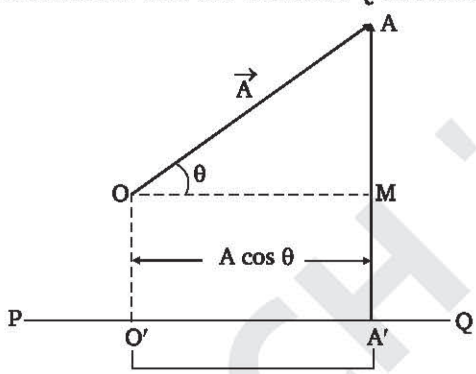


Fig. 1

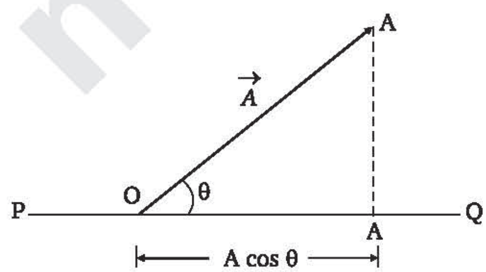


Fig. 2

The length $O'A'$ is called projection of $\vec{OA}$ on line PQ. If vector $\vec{OA}$ makes angle θ with the direction of PQ, then by trigonometry

$$\cos\theta = \frac{OM}{OA}$$

$\Rightarrow$ $$OM = OA\cos\theta$$

$\therefore$ $$O'A' = OM = OA\cos\theta$$

Since the magnitude of $\vec{A} = A$, therefore length $OA = A$.

Thus, the component of vector $\vec{A}$ along line $PQ = A\cos\theta$ ...(1)

From equation (1), it is clear that :

1. Since θ has infinite choice, so a **vector can have infinite components.**
2. Components can have values **positive, zero or negative.**
3. The component of a vector is a scalar so it is specifically called **scalar component.**

If we move vector $\vec{OA}$ in such a way that O and O' coincide, as shown in fig. (2), then the component vector of $\vec{OA}$, along PQ is $\vec{OA'}$ given as :

$$\vec{OA'} = (A\cos\theta)\,\hat{n} \qquad \text{where } \hat{n} \text{ is the unit vector along } PQ.$$

Cartesian Components of a Vector

Cartesian components of a vector are the projections of that vector onto x-, y- and z-axis.

Cartesian components are also called **rectangular components.** For simplicity let us start with two dimensional coordinate system. Consider a vector $\vec{OA} = \vec{A}$, having magnitude A, lying in x-y plane with its tail at origin O [Fig. (3)]. The vector makes angle θ with the direction of positive x-axis. By definition, components of $\vec{A}$ are

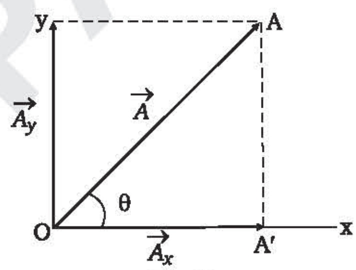


Fig. 3

Component along x-axis (or x-component) $= |\vec{A_x}| = A_x = A\cos\theta$

Component along y-axis (or y-component) $= |\vec{A_y}| = A_y = A\cos\theta$

Vector components of $\vec{A}$ can be written as

Vector x-component $= \vec{A_x} = A_x\,\hat{i}$

Vector y-component $= \vec{A_y} = A_y\,\hat{i}$

(where $\hat{i}$ and $\hat{j}$ are unit vectors along x- and y-axis respectively. From fig. (3), using triangle law

$$\vec{OA} = \vec{OA'} + \vec{A'A}$$

or $$\vec{A} = \vec{A_x} + \vec{A_y} = A_x\,\hat{i} + A_y\,\hat{j} \qquad \text{...(2)}$$

Since the component of vector specify it, eq. (2) is also expressed as

$$\vec{A} = (A_x, A_y) \qquad \text{...(3)}$$

From $\Delta OAA'$ $$(OA)^2 = (AA')^2 + (A'A)^2$$

$$A^2 = A_x^2 + A_y^2$$

$$|\vec{A}| = A = [A_x^2 + A_y^2]^{1/2} \qquad \text{...(4)}$$

and $$\tan\theta = \frac{A_y}{A_x} \qquad \text{...(5)}$$

The above situation can be generalised for vectors having any direction in space. In this case, we introduce z-axis perpendicular to x-y plane, these three mutually perpendicular axis constitutes three dimensional cartesian coordinate system in which a given vector $\vec{A}$ will resolves in 3 components : A_x, A_y and A_z. Then equation (2) can be generalized as :

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \quad ...(6)$$

If vector $\vec{A}$ makes angle α, β, γ respectively with positive directions of x, y and z-axis then :

$$\cos\alpha = \frac{A_x}{A}, \quad \cos\beta = \frac{A_y}{A} \quad \text{and} \quad \cos\gamma = \frac{A_z}{A}$$

are called **direction cosines** of $\vec{A}$.

So, we can show that $\quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \quad ...(7)$

Q.2. What do you mean by scalar product of two vectors? Give its geometrical interpretation. Obtain expression for scalar product of two vectors in terms of their Cartesian components. Mention its important properties and physical significance also.

Ans.

Scalar Product

The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \cdot \vec{B}$ and is read as $\vec{A}$ dot $\vec{B}$. Because of this notation it is also, therefore, known as the *dot product of the two vectors.*

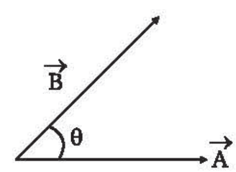


Fig. 1

Mathematically, it is defined as the product of the magnitude of the two vectors $\vec{A}$ and $\vec{B}$ and the cosine of their included angle θ [fig. (1)], irrespective of the coordinate system used. Thus,

$$\vec{A} \cdot \vec{B} = AB\cos\theta \quad ...(1)$$

Since A, B and $\cos\theta$ all are scalars, the right hand side of equation (1) is a scalar quantity. Thus $\vec{A} \cdot \vec{B}$ represents a scalar quantity and their product is called scalar product.

Geometrical Interpretation

A useful geometrical interpretation of scalar product is shown in fig. (2) and fig. (3).

In fig. (2), $B\cos\theta$ is the projection (or the resolute) of $\vec{B}$ in the direction of $\vec{A}$ and in fig. (3), $A\cos\theta$ is the projection (or the resolute) of $\vec{A}$ is the direction of $\vec{B}$.

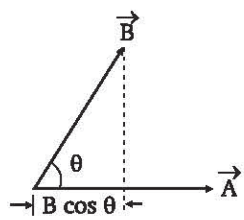


Fig. 2

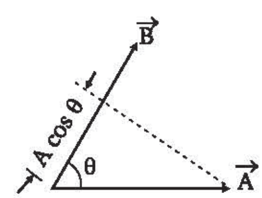


Fig. 3

So, from equation (1) we may write :

$$\vec{A} \cdot \vec{B} = A\,(B\cos\theta) = A\,(\cos\theta)\,B$$

Thus, we may define the scalar product of two vectors ($\vec{A}$ and $\vec{B}$) is the product of the magnitude of either vector and the projection (or the resolute) of the other in its direction.

Scalar product of two vectors in terms of their cartesian components : Let $\vec{A}$ and $\vec{B}$ are two given vectors expressed in terms of their cartesian components as :

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

and

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

then

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= A_xB_x\,\hat{i}\cdot\hat{i} + A_xB_y\,\hat{i}\cdot\hat{j} + A_xB_z\,\hat{i}\cdot\hat{k} + A_yB_y\,\hat{j}\cdot\hat{i}$$

$$+ A_yB_y\,\hat{j}\cdot\hat{j} + A_yB_z\,\hat{j}\cdot\hat{k} + A_zB_x\,\hat{k}\cdot\hat{i} + A_zB_y\,\hat{k}\cdot\hat{j} + A_zB_z\,\hat{k}\cdot\hat{k}$$

...(2)

Since, $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$, therefore in above equation (2) six out of nine terms are zero. So, we have :

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

***i.e.*, scalar product of two vectors in terms of their cartesian components is the sum of products of their respective components.**

Properties of Scalar Product

1. **Depending upon the value of θ, the scalar product of two vectors may be positive, negative or zero.** *i.e.*,

   (a) when $0° \le \theta < 90°$, $\cos\theta$ is +ve, then $\vec{A}\cdot\vec{B}$ is positive

   (b) when $90° \le \theta < 180°$, $\cos\theta$ is –ve, then $\vec{A}\cdot\vec{B}$ is negative

   (c) when $\theta = 90°$, $\cos\theta = 0$, then $\vec{A}\cdot\vec{B}$ is zero.

2. **The scalar product is commutative :** We have,

$$\vec{A}\cdot\vec{B} = AB\cos\theta = BA\cos\theta = \vec{B}\cdot\vec{A} \qquad [\because AB = BA]$$

i.e.,

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

3. **The scalar product is distributive :** In fig. (4), Let $\vec{A}$, $\vec{B}$ and $\vec{C}$ are three arbitrarily chosen vectors, with magnitudes A, B and C respectively.

   The scalar product of vector $\vec{A}$ and the resultant $(\vec{B} + \vec{C})$ of $\vec{B}$ and $\vec{C}$ is given by $\vec{A}\cdot(\vec{B} + \vec{C}) = \vec{A}$ (projection or resolute of $(\vec{B} + \vec{C})$ in the direction of $\vec{A}$), *i.e.,*

$$\vec{A}\cdot(\vec{B}+\vec{C}) = A \times ON = A(OM + MN)$$
$$= A\cdot OM + A\cdot MN. \quad ...(3)$$

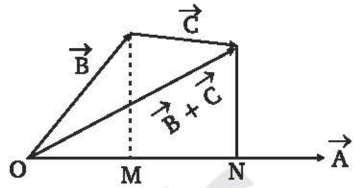


Fig. 4

Since OM is the projection (or resolute) of $\vec{B}$ in the direction of $\vec{A}$ and MN is the projection (or resolute) of $\vec{C}$ in the direction of $\vec{A}$, so we have

$$A\cdot OM = \vec{A}\cdot\vec{B} \quad \text{and} \quad A\cdot MN = \vec{A}\cdot\vec{C}$$

Substituting these values in relation (3) above we have

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

Clearly showing that the **distributive law holds good.**

4. If the two vectors ($\vec{A}$ and $\vec{B}$) be perpendicular to each other, *i.e.*, $\theta = \pi/2$

   Therefore, $\cos\pi/2 = 0$, so, the scalar product $\vec{A}\cdot\vec{B} = AB\cos\left(\frac{\pi}{2}\right) = 0.$

   i.e., $\vec{A}\cdot\vec{B} = 0$

   This is, therefore, the condition for two vectors to be **perpendicular (or orthogonal)** to each other.

5. If the two vectors have opposite directions, $\theta = \pi$ and therefore $\cos\theta = \cos\pi = -1$. So that $\vec{A}\cdot\vec{B} = -AB$, *i.e.*, the scalar product is equal to the negative product of their magnitudes.

6. If the two vectors have the same direction, then $\theta = 0$, therefore $\cos 0° = 1$. So, the scalar product $\vec{A}\cdot\vec{B} = AB\cos\theta = AB\cos 0° = AB$, So, $\vec{A}\cdot\vec{B} = AB$, *i.e.*, **the scalar product is equal to the product of the magnitude of the two vectors.**

   And, if $\vec{A} = \vec{B}$, *i.e.*, if the two vectors be equal, we have

   $$\vec{A}\cdot\vec{A} = |\vec{A}||\vec{A}|\cos 0° = AA = A^2 \text{ or } \vec{A}^2 = A^2,$$

   ***i.e.*, the square of a vector is equal to the square of its magnitude (or modulus).**

7. It follows from (6) above that square of any unit vector, like $\hat{i}$, $\hat{j}$ or $\hat{k}$ is unity.

   Thus, $i^2 = j^2 = k^2 = (1)(1)\cos 0° = 1$

   Or, $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

8. Since $\hat{i}$, $\hat{j}$ and $\hat{k}$ are orthogonal (or perpendicular to each other), therefore, from statement (4) above,

   $$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = (1)(1)\cos 90° = 0.$$

9. If either vector ($\vec{A}$ or $\vec{B}$) is **multiplied by a scalar or a number m,** the scalar product too is multiplied by that number. Thus,

$$(m\vec{A})\cdot\vec{B} = \vec{A}\cdot(m\vec{B}) = m(\vec{A}\cdot\vec{B}) = mAB\cos\theta$$

10. Scalar product give straight forward way to compute angle θ between two vector $\vec{A}$ and $\vec{B}$. As we know,

$$\vec{A}\cdot\vec{B} = |A||B|\cos\theta$$

So,
$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|A||B|}$$

Physical Significance of Scalar Product

Some important applications of scalar product in physics are discussed below :

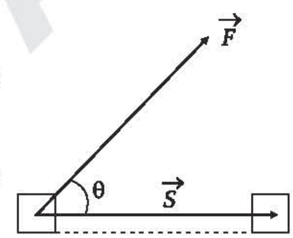


Fig. 5

(a) Work : If a force $\vec{F}$, acting on a body, produces displacement $\vec{S}$. Work done by the force (Fig. 5)

$$W = FS\cos\theta = \vec{F}\cdot\vec{S}$$

(b) Power : Power (defined as rate of doing work) of a force $\vec{F}$ that keeps a body moving with uniform velocity $\vec{v}$ is given by

$$P = \vec{F}\cdot\vec{v}$$

(c) Electric Flux : Electric flux through a planar surface area placed in uniform electric field $\vec{E}$ is given by

$$\phi_E = \vec{E}\cdot\vec{S}$$

Q.3. Define a cross product of two vectors and give a geometrical interpretation of your construction. Obtain expression for cross product of two vectors in terms of their Cartesian components. Mention its important properties and physical significance also.

Ans.

Cross Product of Two Vectors

The vector product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}\times\vec{B}$, read as $\vec{A}$ cross $\vec{B}$. Because of this notation it is, therefore, also called the cross product of the two vectors.

Mathematically, it is defined as a vector ($\vec{R}$) whose magnitude (or modulus) is equal to the product of the magnitudes of the two vectors $\vec{A}$ and $\vec{B}$ and the sine of their included angle θ.

Therefore, $$\vec{R} = \vec{A} \times \vec{B} = |A||B| \sin\theta \, \hat{n} = AB \sin\theta \, \hat{n} \quad ...(1)$$

where A is magnitude of $\vec{A}$, B is magnitude of $\vec{B}$, θ is smaller of the two angle ($\leq 180°$) between $\vec{A}$ and $\vec{B}$, and $\hat{n}$ is a unit vector in the direction of $\vec{A} \times \vec{B}$, (Direction of $\vec{A} \times \vec{B}$ is assigned perpendicular to plane of $\vec{A}$ and $\vec{B}$ such that $\vec{A}, \vec{B}$ and $\vec{A} \times \vec{B}$ form a right handed system.)

Geometrical Interpretation

To obtain $\vec{R}$ geometrically, draw vectors $\vec{A}$ and $\vec{B}$ with their tails coinciding. The two vectors then lie in a plane [Fig. 1]. By definition, vector $\vec{R} = \vec{A} \times \vec{B}$ is perpendicular to plane of $\vec{A}$ and $\vec{B}$. There are always two directions perpendicular to a given plane, one on each side of plane. Which of these two is the direction of $\vec{R}$? To find this, several rules are in use. We describe two of them.

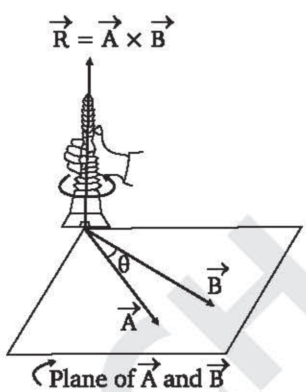


Fig. 1

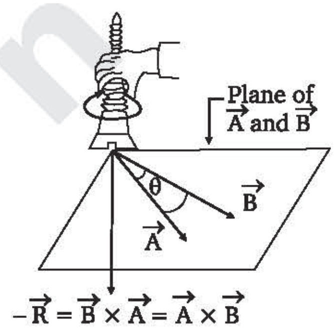


Fig. 2

(a) Right Hand Screw Rule : Place the screw driver with its axis perpendicular to plane containing $\vec{A}$ and $\vec{B}$ [Fig. 1]. Now rotate its handle in the direction from first vector ($\vec{A}$) to second vector ($\vec{B}$) through smaller angle (θ) to coincide with the direction of $\vec{B}$. The direction of advancement of screw is the direction of $\vec{A} \times \vec{B}$.

(b) Right Hand Rule : Stretch your right hand in a plane perpendicular to plane containing $\vec{A}$ and $\vec{B}$, thumb perpendicular to fingers. Now point the fingers along first vector $\vec{A}$. Curl the fingers from $\vec{A}$ towards $\vec{B}$, through smaller angle (θ) to coincide with the direction of $\vec{B}$. Thumb points in the direction of $\vec{R} = \vec{A} \times \vec{B}$ [Fig. 2].

In fig. 2, the vector $\vec{B}$ is rotated towards vector $\vec{A}$ through smaller angle (θ), so the direction of vector $\vec{R}$ is downward to the plane containing $\vec{A}$ and $\vec{B}$.

Expression for cross product of two vectors in terms of their Cartesian components :

We can express vectors $\vec{A}$ and $\vec{B}$ in terms of their Cartesian components as :

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

and,

$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$\therefore$

$$\vec{A} \times \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

$$= A_x B_x \hat{i} \times \hat{i} + A_x B_y \hat{i} \times \hat{j} + A_x B_z \hat{i} \times \hat{k} + A_y B_x \hat{j} \times \hat{i}$$

$$+ A_y B_y \hat{j} \times \hat{j} + A_y B_z \hat{j} \times \hat{k} + A_z B_x \hat{k} \times \hat{i} + A_z B_y \hat{k} \times \hat{j}$$

$$+ A_z B_z \hat{k} \times \hat{k} \quad ...(2)$$

Since, $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$ and $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}, \hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i}$ and $\hat{i} \times \hat{k} = -\hat{j}$, therefore on substituting these values in equation (2), we get three of the nine terms as zero and the six terms that survive give.

$$\vec{A} \times \vec{B} = A_x B_y \hat{k} + A_x B_z (-\hat{j}) + A_y B_x (-\hat{k}) + A_y B_z \hat{i} + A_z B_x \hat{j}$$

$$+ A_z B_y (-\hat{i})$$

Rearranging these terms, we get :

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$

$$= \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \hat{i} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \hat{j} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \hat{k}$$

i.e.,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad ...(3)$$

The determinant in RHS of equation (3) is the most convenient way to remember components of $\vec{A} \times \vec{B}$.

Properties of Vector Product

1. Since $0 \geq \theta \geq \pi$, $\sin\theta$ and, therefore, $|\vec{R}|$ or R cannot be negative.

   And, clearly, $$\sin\theta = \frac{\vec{A} \times \vec{B}}{|\vec{A}||\vec{B}|}$$

2. Vector Product is not commutative

We know that,

$$|\vec{A} \times \vec{B}| = AB\sin\theta$$

Also, $$|\vec{B} \times \vec{A}| = BA\sin\theta$$

∴ $$|\vec{A} \times \vec{B}| = |\vec{B} \times \vec{A}| \qquad (\because AB = BA)$$

So, the vector $\vec{A} \times \vec{B}$ and vector $\vec{B} \times \vec{A}$ have same magnitude, however, their directions are opposite [Fig. 3(a) and 3(b)]. Therefore,

$$(\vec{A} \times \vec{B}) = -(\vec{B} \times \vec{A}) \qquad ...(4)$$

i.e., vector product is **not commutative**

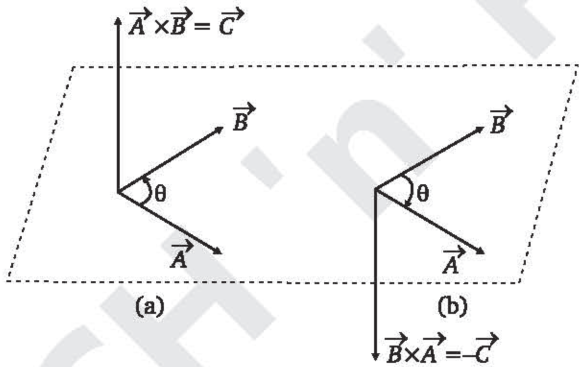


Fig. 3

3. Vector Product obeys distributive law

For three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \qquad ...(5)$$

The geometrical proof of this is tedious. However we shall prove later by Components method.

4. For two collinear (parallel or antiparallel) vectors $\vec{A}$ and $\vec{B}$, $\theta = 0°$ or $180°$, therefore $\sin\theta = 0$

$$|\vec{A} \times \vec{B}| = AB\sin\theta = 0$$

Which gives

$$\vec{A} \times \vec{B} = 0 \quad \text{if } \vec{A} \text{ and } \vec{B} \text{ are collinear}$$

$$\vec{A} \times \vec{A} = \vec{0} \quad \text{for any vector}$$

5. If either vector ($\vec{A}$ or $\vec{B}$) be multiplied by number or a scalar m, their vector product too is multiplied by the same number. Thus,

$$(m\vec{A})\times\vec{B}=\vec{A}\times(m\vec{B})=(mAB)\sin\theta\,\hat{n}.$$

Again $$(m\vec{A})\times(n\vec{B})=(mA\,nB\sin\theta)\hat{n}=n\vec{A}\times m\vec{B}$$

$$=(mn\vec{A})\times\vec{B}=\vec{A}\times(mn\vec{B}),\text{ where } mn \text{ is another scalar.}$$

Thus, we see that the **vector product is associative.**

6. If $\vec{A}$ and $\vec{B}$ are unit vectors, then we have $A=B=1$ and, therefore,

$$\hat{A}\times\hat{B}=AB\sin\theta\,\hat{n}=\sin\theta\,\hat{n},$$

i.e., the magnitude of $\hat{A}\times\hat{B}$ is the sine of the angle of inclination of the two.

7. If the two vectors ($\vec{A}$ and $\vec{B}$) be orthogonal or perpendicular to each other, we have $\theta=90°$ or $\pi/2$ and, therefore, $\sin\theta=1$. So that $\vec{A}\times\vec{B}=AB\hat{n}$.

The vectors $\vec{A},\vec{B}$ and $(\vec{A}\times\vec{B})$ thus form a right-handed system of mutually perpendicular vectors.

8. **For unit vectors of Cartesian Coordinate System :** Let $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along x, y and z-axis. Since the vector product of any vector with itself is a zero vector, so

$$\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0 \quad ...(6)$$

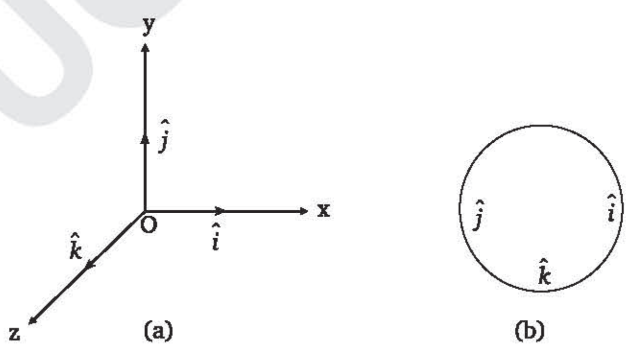


Fig. 4

For right handed Cartesian coordinate system shown in fig.4(a).

$$\hat{i}\times\hat{j}=\hat{k},\quad \hat{j}\times\hat{k}=\hat{i},\quad \hat{k}\times\hat{i}=\hat{j} \quad ...(7)$$

and $$\hat{j}\times\hat{i}=-\hat{k},\quad \hat{k}\times\hat{j}=-\hat{i},\quad \hat{i}\times\hat{k}=-\hat{j} \quad ...(8)$$

A good way to remember eq. (7) and (8) is to write $\hat{i}, \hat{j}, \hat{k}$ cyclic (around a circle) as shown in fig. 5(b). Reading around the circle anti clockwise gives positive product (*e.g.*, $\hat{i} \times \hat{j} = \hat{k}$). Reading other way gives negative results ($\hat{i} \times \hat{k} = -\hat{j}$).

Physical Significance of Vector Product

Vector product has a multitude of applications in physics. Few examples of applications of product are :

(a) Torque : Torque of a force $\vec{F}$ about origin O is defined as

$$\vec{\tau} = \vec{r} \times \vec{F}$$

where $\vec{r}$ is positive vector of point of application of force from O.

(b) Angular Momentum : Angular momentum of a moving particle about a point O is defined as

$$\vec{L} = \vec{r} \times \vec{p}$$

where $\vec{r}$ = position vector of particle from O and $\vec{p}$ = linear momentum

(c) Angular Velocity : If a rigid body rotating with angular velocity $\vec{\omega}$ (which is directed along the axis of rotation), the linear velocity $\vec{v}$ of a point P having position vector $\vec{r}$ relative to fixed origin is given.

$$\vec{v} = \vec{\omega} \times \vec{r}$$

(d) Magnetic Force : Magnetic force on a point charge q moving with velocity $\vec{v}$ in magnetic field $\vec{B}$ is given as

$$\vec{F} = q(\vec{v} \times \vec{B})$$

(e) Area of a Parallelogram : Area of a parallelogram whose adjacent sides represent vectors $\vec{A}$ and $\vec{B}$ is given as

$$\text{Area} = |\vec{A} \times \vec{B}| = AB \sin\theta$$

where θ is the angle between $\vec{A}$ and $\vec{B}$.

Q.4. Define vector triple product of three vectors and obtain its expression in terms of cartesian components. Mention its three important properties and show that associative law does not hold for vector triple product.

Ans.

Vector Triple Product of Three Vectors

Let we have three vectors $\vec{A}, \vec{B}$ and $\vec{C}$ then the quantity $\vec{A} \times (\vec{B} \times \vec{C})$ which yields to a vector quantity is called **vector triple product** of $\vec{A}, \vec{B}$ and $\vec{C}$.

So, for three vectors $\vec{A}, \vec{B}$ and $\vec{C}$, we have $\vec{A} \times (\vec{B} \times \vec{C}), \vec{B} \times (\vec{C} \times \vec{A})$ and $\vec{C} \times (\vec{A} \times \vec{B})$ as examples of vector triple products.

The vector triple product $\vec{A} \times (\vec{B} \times \vec{C})$ is a vector lying in a plane perpendicular to that of $\vec{A}$ and $(\vec{B} \times \vec{C})$. But, as we know, vector $(\vec{B} \times \vec{C})$ lies in the plane normal to that of $\vec{B}$ and $\vec{C}$. It follows, therefore, that vector $\vec{A} \times (\vec{B} \times \vec{C})$ lies in the plane of vector $\vec{B}$ and $\vec{C}$ and is perpendicular to that of $\vec{A}$, as shown in the given figure. Therefore, $\vec{A} \times (\vec{B} \times \vec{C})$ is some vector in the plane of $\vec{B}$ and $\vec{C}$, perpendicular to $\vec{A}$.

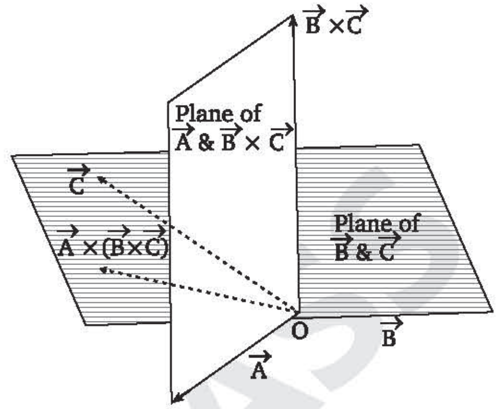


Fig. 1

Vector Product in terms of Cartesian Components

Let us consider vector $\vec{B}$ along *x*-axis, *y*-axis in the plane of $\vec{B}$ and then *z*-axis in the direction of $\vec{B} \times \vec{C}$. Then we have :

$$\left.\begin{aligned} \vec{B} &= B_x \hat{i} \\ \vec{C} &= C_x \hat{i} + C_y \hat{j} \\ \vec{A} &= A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \end{aligned}\right\} \quad ...(1)$$

From this

$$\vec{B} \times \vec{C} = (B_x \hat{i}) \times (C_x \hat{i} + C_y \hat{j}) = B_x C_x (\hat{i} \times \hat{i}) \times B_x C_y (\hat{i} \times \hat{j})$$

$$\Rightarrow \quad \vec{B} \times \vec{C} = B_x C_y \hat{k} \qquad [\text{Since } \hat{i} \times \hat{i} = \vec{0} \text{ and } \hat{i} \times \hat{j} = \vec{k}]$$

Now,

$$\vec{A} \times (\vec{B} \times \vec{C}) = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x C_y \hat{k})$$

$$= A_x B_x C_y (\hat{i} \times \hat{k}) + A_y B_x C_y (\hat{j} \times \hat{k}) + A_z B_x C_y (\hat{k} \times \hat{k})$$

$$= A_x B_x C_y (-\hat{j}) + A_y B_x C_y (\hat{i}) + 0 \quad ...(2)$$

$$[\because \ \hat{k} \times \hat{k} = 0; \ \ \hat{i} \times \hat{k} = -\hat{j}; \ \ \hat{j} \times \hat{k} = \hat{i}]$$

Adding and subtracting $A_x B_x C_x \hat{i}$ in R.H.S. of equation (2), we get

$$\vec{A} \times (\vec{B} \times \vec{C}) = -A_x B_x C_x \hat{i} - A_x B_x C_y \hat{j} + A_x B_x C_x \hat{i} + A_y B_x C_y \hat{i}$$

$$\Rightarrow \quad \vec{A} \times (\vec{B} \times \vec{C}) = -A_x B_x [C_x \hat{i} + C_y \hat{j}] + [A_x C_x + A_y C_y] B_x \hat{i} \qquad \text{...(3)}$$

From set of eq. (1)

$$A_x B_x = \vec{A} \cdot \vec{B} \qquad \text{...(4)}$$

$$A_x C_x + A_y C_y = \vec{A} \cdot \vec{C}$$

Substituting values from eq. (1) and eq. (4) into eq. (3)

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C} \qquad \text{...(5)}$$

Equation (5) is of the form

$$\vec{A} \times (\vec{B} \times \vec{C}) = m \vec{B} + n \vec{C} \qquad \text{...(6)}$$

where *m* and *n* are scalars given as :

$$m = \vec{A} \cdot \vec{C} \qquad \text{and} \qquad n = -(\vec{A} \cdot \vec{B})$$

Thus, $\vec{A} \times (\vec{B} \times \vec{C})$ is a linear combination of two vectors in paranthesis ($\vec{B}$ and $\vec{C}$); the coefficient of each vector is dot product of other two; the middle vector in the product ($\vec{B}$) always has positive sign.

Equation (5) can also be expressed as :

$$\vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \vec{B} & \vec{C} \\ \vec{A} \cdot \vec{B} & \vec{A} \cdot \vec{C} \end{vmatrix} \qquad \text{...(7)}$$

Important Properties of Vector Triple Product

1. The vector triple product $\vec{A} \times (\vec{B} \times \vec{C})$ is a linear combination of those two vectors which are within brackets.
2. The '$\vec{R}$' vector, $\vec{R} = \vec{A} \times (\vec{B} \times \vec{C})$ is perpendicular to vector $\vec{A}$ and remains in the plane of $\vec{B}$ and $\vec{C}$.
3. Vector triple product is recognized as a vector quantity.

Associative law does not hold for vector triple product : We know that, for three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, we have :

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C} \qquad \text{...(8)}$$

Also, for $(\vec{A}\times\vec{B})\times\vec{C} = -\vec{C}\times(\vec{A}\cdot\vec{B})$

$$= -[(\vec{C}\cdot\vec{B})\vec{A} - (\vec{C}\cdot\vec{A})\vec{B}]$$

$$\Rightarrow \quad (\vec{A}\times\vec{B})\times\vec{C} = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{B}\cdot\vec{C})\vec{A} \quad ...(9)$$

From eq. (8) and (9), it is clear that

$$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$$

Thus associative law is not valid for vector triple product in general.

Q.5. Define scalar triple product of three vectors, give its geometrical interpretation and obtain its expression in terms of Cartesian components of vector. Mention its three important properties.

Ans.

Scalar Triple Product of Three Vectors

Consider three non zero vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. We know that cross product of vector $\vec{B}$ and $\vec{C}$ *i.e.*, $\vec{B}\times\vec{C}$ is a vector. Thus $\vec{A}\cdot(\vec{B}\times\vec{C})$ **is a dot product of two vectors $\vec{A}$ and $(\vec{B}\times\vec{C})$** and therefore, gives a scalar. **The product $\vec{A}\cdot(\vec{B}\times\vec{C})$ is called scalar tripple product of vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$.**

$\vec{A}\cdot(\vec{B}\times\vec{C})$ is also denoted as $[\vec{A}\,\vec{B}\,\vec{C}]$

Geometrical Interpretation

Adjoining figure presents a useful geometrical interpretation of scalar triple product. Consider a parallelopiped whose three intersecting edges are represented by vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ having length A, B and C respectively.

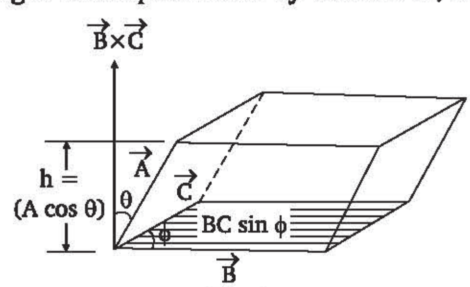


Fig. 1

Area of base of parallelopiped $= BC\sin\phi = |\vec{B}\times\vec{C}|$

Direction of $(\vec{B}\times\vec{C})$ is perpendicular to plane of base. Therefore, height of parallelopiped

$$h = \text{Projection of } \vec{A} \text{ on } (\vec{B}\times\vec{C})$$

or $$h = A\cos\theta = |\vec{A}|\cos\theta$$

Volume of parallelopiped,

$$V = \text{(Area of base) (height } h \text{ of parallelopiped)}$$

$$= |\vec{B}\times\vec{C}||\vec{A}|\cos\theta$$

$$V = \vec{A}\cdot(\vec{B}\times\vec{C}) \quad ...(1)$$

Scalar triple product of three vectors represents the volume of a parallelopiped whose three intersecting edges are represented by these vectors.

If $\theta = 90°$, value of V will come out positive. However, if $\theta > 90°$, value of V will come out negative. Since volume is always positive, we modify eq. (1) as

$$V = |\vec{A} \cdot (\vec{B} \times \vec{C})| = \textbf{modulus of scalar triple product} \quad ...(2)$$

Thus, we see that the scalar triple product $\vec{A} \cdot (\vec{B} \times \vec{C})$ represents the volume of a parallelopiped, with the three vectors forming its three edges.

Obviously, since any one of the faces of the parallelopiped may be taken to be its base, its volume V is also given by $\vec{B} \cdot (\vec{C} \times \vec{A})$ and $\vec{C} \cdot (\vec{A} \times \vec{B})$, with the cyclic order of $\vec{A}, \vec{B}$ and $\vec{C}$ maintained.

In case this cyclic order of the terms in a scalar product is quite immaterial, we have

$$V = \vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{B} \times \vec{C}) \cdot \vec{A} = -\vec{A} \cdot (\vec{C} \times \vec{B}) = -(\vec{C} \times \vec{B}) \cdot \vec{A}$$

$$= \vec{B} \cdot (\vec{C} \times \vec{A}) = (\vec{C} \times \vec{A}) \cdot \vec{B} = -\vec{B} \cdot (\vec{A} \times \vec{C}) = -(\vec{A} \times \vec{C}) \cdot \vec{B}$$

$$= \vec{C} \cdot (\vec{A} \times \vec{B}) = (\vec{A} \times \vec{B}) \cdot \vec{C} = -\vec{C} \cdot (\vec{B} \times \vec{A}) = -(\vec{B} \times \vec{A}) \cdot \vec{C}$$

It will thus be seen that the value of scalar triple product depends on the cyclic order of the vectors and is quite independent of the positions of the dots and the crosses, which may be interchanged as desired. It is, therefore, usual to denote a scalar triple product of vector $\vec{A}, \vec{B}$ and $\vec{C}$ by $[\vec{A}\ \vec{B}\ \vec{C}]$ or $[\vec{A}, \vec{B}, \vec{C}]$, putting the three vectors in their cyclic order but without any dots or crosses.

Scalar triple product in terms of Cartesian components of Vectors

Consider three non-zero vectors $\vec{A}, \vec{B}$ and $\vec{C}$ expressed as :

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$$\vec{C} = C_x \hat{i} + C_y \hat{j} + C_z \hat{k}$$

$$\therefore \qquad \vec{B} \times \vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= (B_y C_z - B_z C_y)\hat{i} - (B_x C_z - B_z C_x)\hat{j} + (B_x C_y - B_y C_x)\hat{k}$$

$$\therefore \qquad \vec{A} \cdot (\vec{B} \times \vec{C}) = A_x (B_y C_z - B_z C_y) - A_y (B_x C_z - B_z C_x) + A_z (B_x C_y - B_y C_x)$$

or $$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Which represents scalar triple product as determinant of rectangular components of vectors.

Properties of Scalar Triple Product

1. If the position of dot and cross is interchanged then there is no change in the value of scalar triple product.

   i.e., $$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$$

2. (a) A cyclic permutation of three vectors does not change the value of scalar triple product *i.e.,*

   $$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

   (b) An anticyclic permutation changes the value of scalar triple product in sign but not in magnitude *i.e.,*

   $$\vec{A}\cdot(\vec{B}\times\vec{C}) = -\vec{A}\cdot(\vec{C}\times\vec{B}) = -\vec{B}\cdot(\vec{A}\times\vec{C}) \text{ and so on}$$

3. If any two vectors in scalar triple product are equal, their scalar triple product is zero.

   i.e., $$[\vec{A}\,\vec{A}\,\vec{C}] = [\vec{C}\,\vec{A}\,\vec{C}] = [\vec{B}\,\vec{B}\,\vec{A}] = 0 \text{ and so on}$$

Q.6. Give some illustrative applications of vector product.

Ans. Some illustrative applications of vector product are :

(i) Torque or moment of a force : Let a force $\vec{F}$ be acting on a body free to rotate about *O* [Fig. 1] and let $\vec{r}$ be the position vector of any point *P* on the line of action of the force. Then, since torque = force × perpendicular distance of its line of action from *O*, we have torque (or moment) of force *F* about *O*.

i.e., $$\tau = F\,r\sin\theta.$$

Now, as we know, $Fr\sin\theta$ is the magnitude of the cross of vector product $\vec{r}\times\vec{F}$. So that, in vector notation, we have

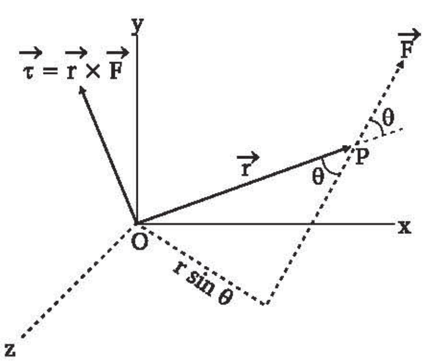


Fig. 1

$$\text{Torque, } \vec{\tau} = \vec{r}\times\vec{F}$$

its direction being perpendicular to the plane containing $\vec{F}$ and $\vec{r}$.

If we draw a set of three coordinate axes through *O*, as shown, we shall have

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},$$

when $x\hat{i}, y\hat{j}$ and $z\hat{k}$ are the rectangular components of $\vec{r}$ along the three axes respectively.

Similarly, $\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$ and $\vec{\tau} = \tau_x \hat{i} + \tau_y \hat{j} + \tau_z \hat{k}$ ...(1)

Now, $$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

which, on expansion, gives $\vec{\tau} = \hat{i}(F_z y - F_y z) + \hat{j}(F_x z - F_z x) + \hat{k}(F_y x - F_x y)$. ...(2)

Comparing expressions (1) and (2), we have

$$\tau_x = F_z y - F_y z, \tau_y = F_x z - F_z x \text{ and } \tau_z = F_y x - F_x y,$$

where τ_x, τ_y and τ_z are the respective scalar components of τ along the three coordinate axes through O.

It will be easily seen that the scalar components τ_x, τ_y and τ_z of torque τ are given by the dot or the scalar products of τ and the respective unit vectors along the three coordinate axes. Thus,

$$\tau_x = \vec{\tau} \cdot \hat{i}, \tau_y = \vec{\tau} \cdot \hat{j} \text{ and } \tau_z = \vec{\tau} \cdot \hat{k}.$$

For $\vec{\tau} \cdot \hat{i} = (\tau_x \hat{i} + \tau_y \hat{j} + \tau_z \hat{k}) \cdot \hat{i} = \tau_x \hat{i} \cdot \hat{i} + \tau_y \hat{j} \cdot \hat{i} + \tau_z \hat{k} \cdot \hat{i}$

And since $\hat{i} \cdot \hat{i} = 1, \hat{j} \cdot \hat{i} = 0$ and $\hat{k} \cdot \hat{i} = 0$, we have $\tau_x = \vec{\tau} \cdot \hat{i}$.

And, similarly, $\tau_y = \vec{\tau} \cdot \hat{j}$ and $\tau_z = \vec{\tau} \cdot \hat{k}$.

(ii) Couple : A couple, as we know, is a combination of two equal, opposite and parallel forces. Let $\vec{F}$ and $-\vec{F}$ be two such forces acting at points P and Q [fig.2] and let the position vectors of P and Q with respect to O be $\vec{r_1}$ and $\vec{r_2}$ respectively.

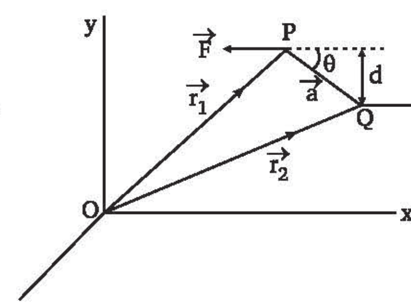


Fig. 2

Then, since moment of the couple $\vec{C}$ with respect to O is equal to the sum of the moments, (with respect to O) of the two forces constituting the couple, we have

$$\vec{C} = \vec{r_1} \times \vec{F} + \vec{r_2} \times (-\vec{F}) = (\vec{r_1} - \vec{r_2}) \times \vec{F}$$

Since $\vec{r_1} - \vec{r_2} = \vec{a}$, where $\vec{a}$ lies in the same plane with $\vec{F}$, we have $\vec{C} = \vec{a} \times \vec{F}$

Thus, the moment of the couple $\vec{C}$, is a vector lying in a plane perpendicular to that containing the two forces.

The magnitude of couple $\vec{C} = |\vec{a} \times \vec{F}|$ or $aF \sin\theta$.

And since $a \sin\theta = d$, the perpendicular distance between the two forces, we have magnitude of the moment of the couple, *i.e.*, $C = Fd$ = (one of the forces)(perpendicular distance between the forces).

(iii) The law of sines in a triangle : Consider a triangle of vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ [Fig. 3], such that $\vec{A} + \vec{B} = \vec{C}$.

Taking vector product of both sides of the relation with $\vec{A}$, we have $\vec{A} \times \vec{A} + \vec{A} \times \vec{B} = \vec{A} \times \vec{C}$.

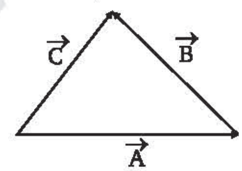


Fig. 3

Or, since $\vec{A} \times \vec{A} = 0$

$\therefore$ we have $\qquad \vec{A} \times \vec{B} = \vec{A} \times \vec{C}$.

So that, taking magnitudes of the two sides, we have

$$AB \sin(\vec{A} \cdot \vec{B}) = AC \sin(\vec{A} \cdot \vec{C})$$

where $\qquad \sin(\vec{A}, \vec{B})$ = sine of the angle between $\vec{A}$ and $\vec{B}$

and $\qquad \sin(\vec{A}, \vec{C})$ = sine of the angle between $\vec{A}$ and $\vec{C}$

Or $\qquad B \sin(\vec{A}, \vec{B}) = \vec{C} \sin(\vec{A}, \vec{C})$,

hence, $\qquad \dfrac{B}{\sin(\vec{A}, \vec{C})} = \dfrac{C}{\sin(\vec{A}, \vec{B})}$

which is the familiar law of sines in a triangle.

(iv) Force on a moving charge in a magnetic field : Imagine a charge q to be moving with velocity $\vec{v}$ at an angle θ with a magnetic field $\vec{B}$ at any given instant [Fig. 4]. Then, the force acting on it in a direction perpendicular to $\vec{B}$ as well as $\vec{v}$ is $F = qvB \sin\theta$, where F, v and B are the magnitudes of the force velocity and the magnetic field respectively. In vector form, therefore, we may put it as

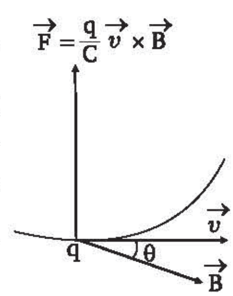


Fig. 4

$$\vec{F} = q(\vec{v} \times \vec{B}) \text{ emu or SI units.}$$

This is called Lorentz force law, with the force itself referred to as the Lorentz force.

In most cases, q is taken in esu and $|\vec{B}|$ in emu (*i.e.*, gauss). Since 1 esu charge $= (1/c)$ emu, when c is the velocity of light in vacuo, we have q *esu* $= q/c$ emu and, therefore,

$$\vec{F} = \frac{q}{c}(\vec{v} \times \vec{B})$$

In case the charge also simultaneously passes through an electric field $\vec{E}$, an additional force $q\vec{E}$ acts upon it and the Lorentz force law then takes the form

$$\vec{F} = q\vec{E} + \frac{q}{e}(\vec{v} \times \vec{B})$$

(v) Area of a parallelogram : Let vectors $\vec{A}$ and $\vec{B}$ from the adjacent sides of a parallelogram $OPQR$, [Fig. 5] inclined to each other at an angle θ. Then, if OD be the perpendicular dropped from O on to PQ, we have area of the parallelogram

$= PQ\,(OD) = B\,(A \sin\theta) = AB \sin\theta$

which is obviously twice the area of the triangle OPQ with the same adjacent sides $\vec{A}$ and $\vec{B}$.

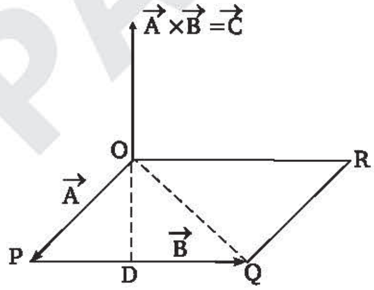


Fig. 5

Clearly, $AB \sin\theta$ is the magnitude of the vector product $\vec{A} \times \vec{B} = \vec{C}$, say, whose direction is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$, *i.e.*, to the plane of the parallelogram.

Now, an area, by itself, has no sign but may be regarded positive or negative in relation to the direction in which its boundary is described. $\vec{A} \times \vec{B} = \vec{C}$, therefore, represents a vector area which gives both the magnitude and the orientation of the area of the parallelogram.

The direction of vector $\vec{C}$, drawn normal to the plane of the figure (*i.e.*, the parallelogram here) bears the same relation to the direction in which the boundary of the figure is described as the direction of advance of a right-handed screw does to its direction of rotation. Thus, with the parallelogram described as shown ($\vec{A}$ being taken first and $\vec{B}$ next), the normal vector $\vec{C} = \vec{A} \times \vec{B}$ is drawn pointing upwards, the area of the parallelogram being regarded as positive in relation to this direction of $\vec{C}$.

With this convention, therefore, any vector area (*i.e.*, the area of any plane figure) may be represented by a vector drawn normal to the plane of the figure in a direction relative to which the area of the figure is regarded as positive.

❑

UNIT-II

Vector Calculus

SECTION-A VERY SHORT ANSWER TYPE QUESTIONS

Q.1. Define integration.

Ans. Integration of $f(x)$ with the respect to x is defined as

$$I = \int f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1} f(x_i)\,\Delta x \text{ as } x \to 0, N \to \infty$$

This is called indefinite integral. Definite integral is denoted as

$$I = \int_a^b f(x)\,dx$$

Q.2. If $\vec{A}$ is differentiable function of scalar variable t, show that

$$\frac{d}{dt}\left(\vec{A} \times \frac{d\vec{A}}{dt}\right) \Rightarrow A \times \frac{d^2\vec{A}}{dt^2}$$

Sol.

$$\frac{d}{dt}\left(\vec{A} \times \frac{d\vec{A}}{dt}\right) = \vec{A} \times \frac{d^2\vec{A}}{dt^2} + \frac{d\vec{A}}{dt} \times \frac{d\vec{A}}{dt}$$

$$= \vec{A} \times \frac{d^2\vec{A}}{dt^2} + \vec{0} \qquad \left[\because \frac{d\vec{A}}{dt} \times \frac{d\vec{A}}{dt} = \vec{0}\right]$$

$$= A \times \frac{d^2\vec{A}}{dt^2}$$

Proved.

Q.3. Defined a scalar field.

Ans. Scalar Field : A region of space in which a scalar quantity ϕ can be expressed as continuous single valued function of position *i.e.*, $\phi = \phi\ (x, y, z)$.

Examples : Distribution of temperature $T\ (x, y, z)$ in a given region of space or electrostatic potential $V\ (x, y, z)$ in the region of space around a charged body or gravitational potential $V\ (x, y, z)$ around earth.

Q.4. What is Laplacian operator?

Ans. The Laplacian operator is defined as $v^2 = \partial z\, \partial \times 2 + \partial z\, \partial yz + \partial z\, \partial z2$.

The Laplacian is a scalar operator. It is applied to a scalar field it generates as scalar field.

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ is known as **Laplacian operator**.

Q.5. Defined a vector field.

Ans. Vector Field : A region space in which a vector quantity $\vec{A}$ can be expressed as a continuous single valued function of position *i.e.*, $\vec{A} = \vec{A}(x, y, z)$.

Example : Distribution of electric field $\vec{E}\,(x, y, z)$ in a region surrounding a charged body, distribution of magnetic field $\vec{B}\,(x, y, z)$ in a region around a magnet or steady current etc. Physical problems are often restricted to certain region of space and our mathematics must take it into account.

Q.6. What are solenoidal field and irrotational field?

Ans. A **solenoidal vector field** is known as an incompressible vector field of which divergence is zero. Hence, a solenoidal vector field is called a divergence-free vector field. On the other hand, an **irrotational vector field** implies that the value of curl at any point of the vector field is zero.

Q.7. Define conservation field.

Ans. Conservative Field : If the value of line integral of a vector point function depend only upon initial and final points and not upon the actual path taken, the vector field is referred to as **Conservative field.**

Example : Gravitational field, Electrostatic field etc. For conservative field :

1. $\int_{P_1}^{P_2} \vec{A} \cdot \vec{dl} \rightarrow$ path independent.
2. $\oint \vec{A} \cdot \vec{dl} =$ line integral around close curve $= 0$.
3. curl $\vec{A} = 0$.

Q.8. Determine the constant a so that the vector $\vec{A} = (-4x - 6y + 3z)\hat{i} + (-2x + y - 5z)\hat{j} + (5x + 6y + az)\hat{k}$ is solenoidal.

Sol. Vector $\vec{A}$ is solenoidal, hence

$$\text{div}\,\vec{A} = 0$$

$$\therefore \quad \frac{\partial Ax}{\partial x} + \frac{\partial Ay}{\partial y} + \frac{\partial Az}{\partial z} = 0$$

$$\therefore \quad \frac{\partial}{\partial x}(-4x - 6y + 3z) + \frac{\partial}{\partial y}(-2x + y - 5z) + \frac{\partial}{\partial z}(5x + 6y + az) = 0$$

$$\therefore \quad -4 + 1 + a = 0$$

$$a = 3$$

Q.9. What do you mean by differentiation of vectors?

Ans.

Differentiation of Vector

If we have a quantity y whose value depends upon single variable x, relation is expressed as

$$y = f(x) \qquad \text{[Real single valued function]}$$

If $x \to x+\Delta x$ then $y \to y+\Delta y$, we may write

$\therefore$ $$y+\Delta y = f(x+\Delta x)$$

Derivative of y with respect to x is defined as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$\frac{dy}{dx}$ represents slope of $x - y$ curve at $y = x$. It is basically a rate measurer.

Q.10. Evaluate $\int_{-\infty}^{\infty} (2x^2 - 3x + 4)\delta(x-2)\,dx$.

Sol. By property of Dirac delta function

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = fa$$

Here $$f(x) = 2x^2 - 3x + 4 \text{ and } a = 2$$

$$\text{Hence the integral} = (2x^2 - 3x + 4)_{x=2}$$

$$= 2(2)^2 - 3(2) + 4 = 8 - 6 + 4 = 6.$$

Q.11. If $\vec{r} = e^{nt}\vec{a} + e^{-nt}\vec{b}$ where $\vec{a}$ and $\vec{b}$ are constant vectors, show that

$$\frac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0$$

Sol. $$\frac{d\vec{r}}{dt} = ne^{nt}\vec{a} - ne^{-nt}\vec{b}$$

$$\frac{d^2\vec{r}}{dt^2} = n^2e^{nt}\vec{a} + n^2e^{-nt}\vec{b}$$

$$= n^2(e^{nt}\vec{a} + e^{-nt}\vec{b}) = n^2\vec{r}$$

$\therefore$ $$\frac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0$$ **Proved**

Q.12. Give two examples of physical quantities in physics which are expressed as gradient of some scalar point function.

Ans. Two examples of Gradient in Physics are as follows :

1. In a conservative field, force $(\vec{F})$ is represented as gradient of potential energy (U)

$$\vec{F} = -\vec{\nabla}U$$

equation holds for both gravitational and electrostate field.

2. Electrostatic field $\vec{E}$ is represented as gradient of electrostatic potential (V) *i.e.*,

$$\vec{E} = -\vec{\nabla}V$$

SECTION-B SHORT ANSWER TYPE QUESTIONS

Q.1. Define partial derivatives of a vector.

Ans.

Partial Derivatives of a Vector

Suppose a vector $\vec{A}$ depends upon more than one variable, say x, y, z. We may write

$$\vec{A} = \vec{A}(x, y, z)$$

The differentiation of such vector with respect to any one variable, keeping other variable constant, is called **partial differentiation**. The derivatives of vector so obtained are called **partial derivatives** and are denoted by symbol ∂ instead of d.

For vectors $\vec{A}(x, y, z)$ partial derivatives can be expressed as :

$$\frac{\partial \vec{A}}{\partial x} = \lim_{\Delta x \to 0} \frac{\vec{A}(x+\Delta x, y, z) - \vec{A}(x, y, z)}{\Delta x}$$

$$\frac{\partial \vec{A}}{\partial y} = \lim_{\Delta y \to 0} \frac{\vec{A}(x, y+\Delta y, z) - \vec{A}(x, y, z)}{\Delta y}$$

$$\frac{\partial \vec{A}}{\partial z} = \lim_{\Delta z \to 0} \frac{\vec{A}(x, y, z+\Delta z) - \vec{A}(x, y, z)}{\Delta z}$$

Higher derivatives can be defined as in calculus. For example

$$\frac{\partial^2 \vec{A}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{A}}{\partial x}\right), \frac{\partial^2 \vec{A}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{A}}{\partial y}\right), \frac{\partial^2 \vec{A}}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \vec{A}}{\partial z}\right)$$

$$\frac{\partial^2 \vec{A}}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{A}}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \vec{A}}{\partial x}\right) \text{ etc.}$$

Order of differentiation does not matter.

Q.2. If $\phi(x, y, z) = 3x^2 y - y^2 z^2$. Find grad ϕ at $(1, -2, -1)$.

Sol.

$$\text{grad}\,\phi = \vec{\nabla}\phi$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2 y - y^2 z^2)$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2 y - y^2 z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2 y - y^2 z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2 y - y^2 z^2)$$

$$= 6xy\,\hat{i} + (3x^2 - 2yz^2)\,\hat{j} + 2y^2 z\,\hat{k}$$

$$\vec{\nabla}\phi\,(1, -2, 1) = 6(1)(-2)\,\hat{i} + [3(1)^2 - 2(-2)(1)^2]\,\hat{j} + 2(-2)^2(1)\,\hat{k}$$

$$= \mathbf{-12\,\hat{i} + 7\,\hat{j} + 8\,\hat{k}}$$

Q.3. Define the operator $\vec{\nabla}$.

Ans.

The Operator $\vec{\nabla}$

In vector calculus, a differential operator $\vec{\nabla}$, called del, is defined as

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$\vec{\nabla}$ is not a vector. When we write $\vec{\nabla}\phi$, $\vec{\nabla}$ is not multiplied by scalar ϕ. Rather it is a **vector operator** that acts upon ϕ. It is more complicated than $\frac{d}{dx}$ (which is scalar operator) because $\vec{\nabla}$ has vector properties too.

Now, an ordinary vector $\vec{A}$ can multiply itself in three ways :

1. Multiply a scalar ϕ : $\phi\vec{A}$ [Result is a vector]
2. Multiply another vector $\vec{B}$: $\vec{A}\cdot\vec{B}$ [Result is a vector] via dot product
3. Multiply anothe vector $\vec{B}$: $\vec{A}\times\vec{B}$ (Result is a vector) via cross product

Correspondingly there are three ways the operator $\vec{\nabla}$ can act :

1. On scalar function ϕ : $\vec{\nabla}\phi$ (the gradient, which is a vector).
2. One vector function $\vec{A}$: $\vec{\nabla}\cdot\vec{A}$ (the divergence, which is a scalar) via the dot product
3. On vector function $\vec{A}$: $\vec{\nabla}\times\vec{A}$ (the curl, which is a vector) via the cross product

We have already discussed gradient of scalar function. In the following sectiosn we shall explain divergence and curl of vector function.

Q.4. Define curl of a vector field. Gives its geometrical interpretation.

Ans.

Curl of a Vector Field

Consider a vector field in which $\vec{A}$ is a continuously differentiable vector point function at each point (x, y, z), expressed as

$$\vec{A}(x, y, z) = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

Curl of $\vec{A}$ is defined as

$$\text{Curl }\vec{A} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

It is clear that curl $\vec{A}$ is a vector. If we expand the determinant, we get

$$\text{Curl } \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{A} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_z}{\partial y}\right)\hat{k}$$

Thus Curl $\vec{A}$ is a vector whose Cartesian components are :

$$[\text{Curl } \vec{A}]_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$[\text{Curl } \vec{A}]_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

$$[\text{Curl } \vec{A}]_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

Geometrical Interpretation

As the name **Curl** implies $\vec{\nabla} \times \vec{A}$ is a measure of how much vector $\vec{A}$ curls around given point. Four vector functions have zero curl. However, the vector function shown in figure 1 has substantial curl pointing in z direction according to natural right hand rule as the vector lies in $x - y$ plane and circulating counter clockwise.

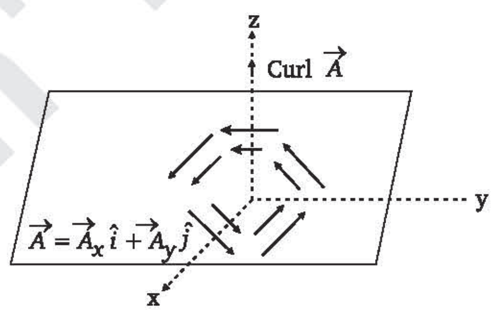


Fig. 1

The term **rotation** is also sometimes used for **curl.** A vector field for which curl is zero, is called **irrotational** *i.e.*,

(i) If curl $\vec{A} = \vec{0}$, $\vec{A}$ is irrotational

(ii) If curl $\vec{A} \neq 0$, $\vec{A}$ is rotational.

Fig. 2 (a) and 2 (b) show electrostatic field line due to a –ve charge and +ve charge respectively. Fig 2 (c) and 2(d) show magnetic field line due to a current carrying wire and bar magnet respectively. From this figures it is clear that :

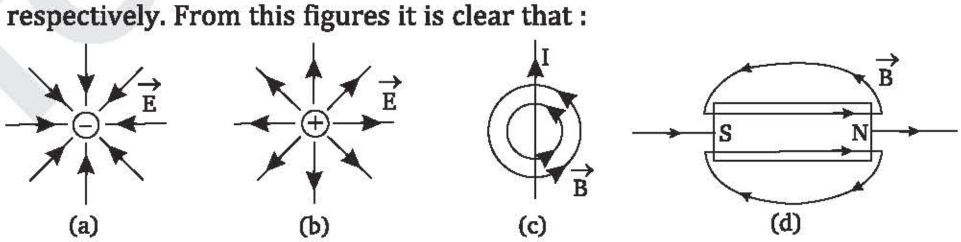


Fig. 2

(a) Electronic field $\vec{E}$ is irrotation *i.e.*, curl $\vec{E} = 0$.

(b) Magnetic field $\vec{B}$ is rotational *i.e.*, curl $\vec{B} \neq 0$.

Physical interpretation of curl will be discussed later in this chapter after the introduction of vector integration.

Q.5. Define line integral of a vector. Give one example.

Ans.

Line Integrals

Integration of a vector along a line or curve is called **line integral.**

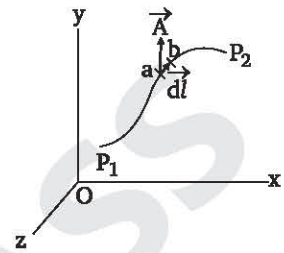


Fig.

Let $\vec{A}(x, y, z)$ be a vector function defined in some region of space (fig.).

Consider a curve drawn from P_1 to P_2 in a vector field, along which vector $\vec{A}$ is to be integrated. $\vec{A}$ is continuous function along the curve. If $\vec{dl}$ is element of length on this curve drawn from point $a\,(x, y, z)$ to $b\,(x+dx, y+dy, z+dz)$, then we can express infinitesimal displacement vector $\vec{dl}$ as :

$$\vec{dl} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

If $\vec{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$ is the value of vector function on this element $\vec{dl}$, then line integral of $\vec{A}$ along the curve P_1P_2 is defined as

$$\text{Line Integral} = \int_{P_1}^{P_2} \vec{A}\cdot\vec{dl} = \int_{P_1}^{P_2} A\,dl\cos\theta \qquad \text{...(1)}$$

In terms of components of vector

$$\int_{P_1}^{P_2} \vec{A}\cdot\vec{dl} = \int_{P_1}^{P_2} (Ax\,dx + Ay\,dy + Az\,dz) \qquad \text{...(2)}$$

Line integral around a close curve is expressed as $\oint \vec{A}\cdot\vec{dl}$.

Examples : (i) Workdone by a variable force $\vec{F} = \vec{F}\,(x, y, z)$ in moving an object from initial position $P_1(x_1, y_1, z_1)$ to final position $P_2\,(x_2, y_2, z_2)$ is calculated as

$$dW = \vec{F}\cdot\vec{dl}$$

$$W = \int_{P_1}^{P_2} \vec{F}\cdot\vec{dl} = \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy + \int_{z_1}^{z_2} F_z dz$$

(ii) Electrostatic potential difference between two points a and b in electrostatic field is calculated as

$$V_b - V_a = -\int_a^b \vec{E}\cdot\vec{dl}$$

Q.6. If $\phi = \ln|\vec{r}|$, find $\vec{\nabla}\phi$.

Sol. $\therefore$ $\quad \vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

$\therefore$ $\quad |\vec{r}| = (x^2 + y^2 + z^2)^{1/2}$

$$\therefore \qquad \phi = \ln|\vec{r}| = \ln(x^2+y^2+z^2)^{1/2} = \frac{1}{2}\ln(x^2+y^2+z^2)$$

$$\therefore \qquad \vec{\nabla}\phi = \frac{1}{2}\left[\hat{i}\frac{\partial}{\partial x}\ln(x^2+y^2+z^2)+\hat{j}\frac{\partial}{\partial y}\ln(x^2+y^2+z^2)+\hat{k}\frac{\partial}{\partial z}\ln(x^2+y^2+z^2\right]$$

$$= \frac{1}{2}\left[\hat{i}\left(\frac{2x}{x^2+y^2+z^2}\right)+\hat{j}\left(\frac{2y}{x^2+y^2+z^2}\right)+\hat{k}\left(\frac{2z}{x^2+y^2+z^2}\right)\right]$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)} = \frac{\vec{\mathbf{r}}}{\mathbf{r}^2}$$

Q.7. Define volume integral giving one example.

Ans.

Volume Integral

Volume integrals are somewhat simpler since element of volume dV is a scalar.

Consider a continuous vector function $\vec{A}(x, y, z)$ in space. Consider a small region of volume V (fig). V can be divided into large number of infinitesimal elements of volume dV inside which $\vec{A}$ can be assumed constant.

Volume integral of $\vec{A}$ over volume V is defined as

$$\text{Volume integral} = \int_{\text{Volume}} \vec{A}\, dV = \iiint \vec{A}\, dV \qquad ...(1)$$

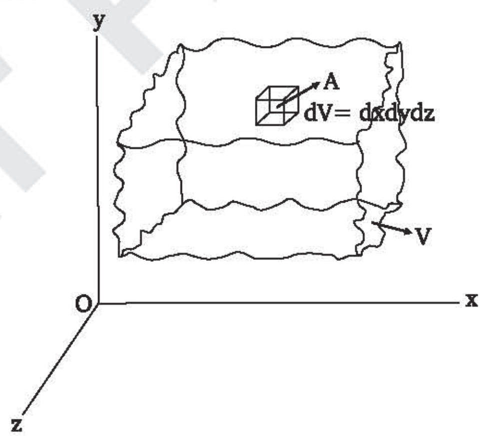


Fig.

In terms of Cartesian components

$$\vec{A} = Ax\,\hat{i} + Ay\,\hat{j} + Az\,\hat{k} \text{ and } dV = dx\,dy\,dz$$

$$\text{Volume integral} = \hat{i}\iiint Ax\,dx\,dy\,dz + \hat{j}\iiint Ay\,dx\,dy\,dz + \hat{k}\iiint Az\,dx\,dy\,dz \qquad ...(2)$$

For a scalar point function $\phi(x, y, z)$ volume integral

$$= \iiint \phi\, dx\,dy\,dz \qquad ...(3)$$

Example : In eq. (3) $\phi = \phi\,(x, y, z) =$ density of a substance mass

$$M = \iiint \phi\,(x, y, z)\,dV$$

Q.8. Define surface integral, giving one example.

Ans.

Surface Integral

Consider the field of a continuous vector point function $\vec{A}(x, y, z)$. Let S is an open surface of arbitrary shape. We can divide it into large number of infinitesimal plane element of area dS.

For such element, area vector $\vec{dS}$ is defined as a vector of magnitude dS, directed along outward normal. (Fig.)

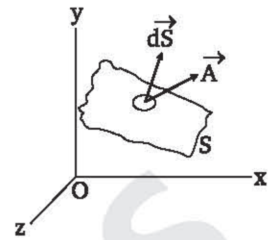


Fig.

If $\vec{A}$ is a value of vector point function at centre of $\vec{dS}$, the quantity.

$$\vec{A}\cdot\vec{dS} = \text{flux of } \vec{A} \text{ through } \vec{dS} = d\phi$$

$\int \vec{A}\cdot\vec{dS}$ over entire surface is called surface integral of $\vec{A}$ over surface S.

It represents flux of $\vec{A}$ through entire surface S.

Since the surface integral is a two dimensional quantity, it is represented by double integral *i.e.*,

$$\int_S \vec{A}\cdot\vec{dS} = \iint \vec{A}\cdot\vec{dS} = \phi \text{ (Flux)} \qquad ...(1)$$

In terms of Cartesian components.

$$\vec{A} = Ax\,\hat{i} + Ay\,\hat{j} + Az\,\hat{k}$$

$$\vec{dS} = \hat{i}\,dS_x + \hat{j}\,dS_y + \hat{k}\,dS_z$$

$$\therefore \qquad \iint \vec{A}\cdot\vec{dS} = \iint (Ax\,dS_x + Ay\,dS_y + Az\,dS_z) \qquad ...(2)$$

Example : (i) Electric flux through a surface in electric field is calculated as :

$$\phi_E = \iint \vec{E}\cdot\vec{dS}$$

(ii) Magnetic flux through a surface in magnetic field is given by

$$\phi_B = \iint \vec{B}\cdot\vec{dS}$$

Q.9. State the Greens theorem and write properties.

Ans.

Green's Theorem

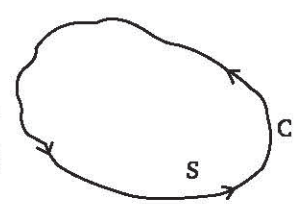


Fig.

The theorem may be stated as follows :

"If S is a closed region in $X-Y$ plane bounded by a simple close curve C (fig) and P and Q are continuous functions of x and y having continuous derivatives in regions, then"

$$\oint_C (P\,dx + Q\,dy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where line integral is counter clockwise around the boundary of areas.

This is called Green's Theorem in the plane.

Properties

1. Green theorem in the plane is a special case of Stoke's theorem.
2. Gauss divergence theorem is a generalization to Green's theorem in the plane where the (plane) region S and its closed boundary (curve) C are replaced by a (space) region V and its closed boundary (surface) S respectively. Due to this, Gauss divergence theorem is often called **Green's theorem in space.**
3. Green theorem defined as above do not work for region S which have holes in them. For such regions also, theorem may be stated in a different manner.

Q.10. If $\phi = 6x^3 y^2 z$, find div grad ϕ.

Sol.

$$\text{grad}\,\phi = \vec{\nabla}\phi$$
$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$
$$= \hat{i}\frac{\partial}{\partial x}(6x^3y^2z) + \hat{j}\frac{\partial}{\partial y}(6x^3y^2z) + \hat{k}\frac{\partial}{\partial z}(6x^3y^2z)$$
$$= 18x^2y^2z\,\hat{i} + 12x^3yz\,\hat{j} + 6x^3y^2\,\hat{k}$$

$$\text{div grad}\,\phi = \vec{\nabla}\cdot\vec{\nabla}\phi$$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(18x^2y^2z\,\hat{i} + 12x^3yz\,\hat{j} + 6x^3y^2\,\hat{k})$$
$$= \frac{\partial}{\partial x}(18x^2y^2z) + \frac{\partial}{\partial y}(12x^3yz) + \frac{\partial}{\partial z}(6x^3y^2)$$
$$= \mathbf{36xy^2z + 12x^3z.}$$

Q.11. State Helmholtz theorem.

Ans.

Helmholtz's Theorem

A vector is uniquely specified by giving its divergence and curl within the specified region (without holes)and its normal component over the boundary.

Consider a vector field $\vec{F}$

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{F} &= S = \text{Source (charge) density} \\ \vec{\nabla}\times\vec{F} &= C = \text{Circulation (current) density} \end{aligned}\right\} \quad ...(1)$$

Helmholtz theorem states :

"A vector $\vec{F}$ satisfying eq. (2) with both source and circulation densities vanishing at infinity may be expressed as sum of two parts, one of which is irrotational, the other part is solenoidal."

$$\vec{F} = -\underbrace{\vec{\nabla}\phi}_{\text{irrotational}} + \underbrace{\vec{\nabla}\times\vec{A}}_{\text{solenoidal}} \quad ...(2)$$

Here,

I part = grad of scalar function ϕ, ϕ is called **scalar potential.**

Since curl grad $\phi = \vec{\nabla}\times(\vec{\nabla}\vec{A}) = 0$, $\vec{\nabla}\phi$ is irrotational.

II part = curl of vector function $\vec{A}$, $\vec{A}$ is called **vector potential.**

Since, div curl $\vec{A} = \vec{\nabla}\cdot(\vec{\nabla}\phi) = 0$, $\vec{\nabla}\times\vec{A}$ is solenoidal.

SECTION-C LONG ANSWER TYPE QUESTIONS

Q.1. What do you mean by derivative of a vector? Gives its geometrical interpretation. Obtain expression for derivative of a vector in terms of cartesian components.

Ans.

Vector Derivative

Definition : Vector quantities are often expressed as functions of scalar variable. Consider a vector $\vec{A}$ given as

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_2 \hat{k} \qquad ...(1)$$

Here $\hat{i}, \hat{j}, \hat{k}$ are fixed unit vectors. If A_x, A_y and A_z are functions of a single scalar variable t, we may write

$$\left.\begin{aligned} A_x &+ A_x(t) \\ A_y &= A_y(t) \\ A_z &= A_z(t) \end{aligned}\right] \text{ scalar functions of scalar variable } t$$

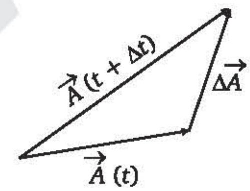


Fig. 1

Hence, vector $\vec{A}$ is also function of scalar variable t. We may write

$$\vec{A} = \vec{A}(t) \text{ vector function of scalar variable } t$$

Consider vector function $\vec{A}(t)$. The change in $\vec{A}$ during interval from t to $t + \Delta t$ is (Fig. 1)

$$\Delta \vec{A} = \vec{A}(t + \Delta t) - \vec{A}(t)$$

The derivative of $\vec{A}$ with respect to t is defined as

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{A}}{\Delta t} = \lim_{t \to 0} \frac{\vec{A}(t + \Delta t) - \vec{A}(t)}{\Delta t} \qquad ...(2)$$

$\frac{d\vec{A}}{dt}$ is a new vector which can be large or small. Direction of $\frac{d\vec{A}}{dt}$ can be any, depending upon behaviour of $\vec{A} \cdot \frac{d\vec{A}}{dt}$ points in direction of $\Delta \vec{A}$ as ΔA approaches zero.

Applying eq. (2) in eq. (1), we obtain

$$\frac{d\vec{A}}{dt} = \hat{i}\frac{dA_x}{dt} + \hat{j}\frac{dA_y}{dt} + \hat{k}\frac{dA_z}{dt} \qquad ...(3)$$

Geometrical Interpretation of Vector Derivative

Let (x, y, z) be the coordinates of a moving particle at time t; then x, y, z are function of t *i.e.*,

$$x = x(t), \;\; y = y(t), \;\; z = z(t) \qquad ...(1)$$

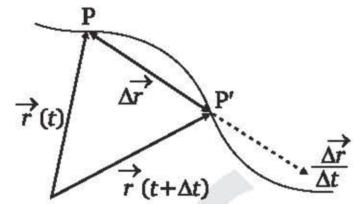


Fig. 2

Displacement of particle from origin at time t is

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \qquad ...(2)$$

As t changes, terminal point of $\vec{r}$ describes a space curve having parametric equation defined as equation (1). Leter, at time $t + \Delta t$, position vector of particle is

$$\vec{r}(t+\Delta t) = x(t+\Delta t)\hat{i} + y(t+\Delta t)\hat{j} + z(t+\Delta t)\hat{k} \qquad ...(3)$$

The change in $\vec{r}$ in interval t to Δt is

$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$$

$$\therefore \qquad \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \qquad ...(4)$$

If P and P' are the positions of particle at time t and $t + \Delta t$ respectively, as shown in fig 2 since $\Delta\vec{r}$ is a vector, the quotient $\frac{\Delta\vec{r}}{\Delta t}$ is also a vector. Direction of $\frac{\Delta\vec{r}}{\Delta t}$ is along chord PP' from P to P'.

As Δt tends to zero $(\Delta t \to 0)$:

(i) P' approaches P and chord PP' tends to concide with the tangent to curve at point P.

(ii) $\underset{\Delta t \to 0}{\text{Limit}} \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta\vec{r}}{dt}$.

Thus, $\frac{\Delta\vec{r}}{dt}$ is a vector in the direction of tangent to space curve $(\vec{r}\ vs\ t)$ at position $\vec{r}$ defined by coordinates (x, y, z).

Physical Significance

Derivatives of vector quantity may lead to another vector quantity. If we put value of $\vec{r}(t)$ and $\vec{r}(t+\Delta t)$ from equations (2) and (3) in equation (4).

$$\frac{\Delta\vec{r}}{\Delta t} = \left\{\frac{x(t+\Delta t) - x(t)}{\Delta t}\right\}\hat{i} + \left\{\frac{y(t+\Delta t) - y(t)}{\Delta t}\right\}\hat{j} + \left\{\frac{z(t+\Delta t) - z(t)}{\Delta t}\right\}\hat{k}$$

Taking limit as $\Delta t \to 0$

$$\frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}\right)\hat{i} + \left(\frac{dy}{dt}\right)\hat{j} + \left(\frac{dz}{dt}\right)\hat{k}$$

or

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \qquad ...(5)$$

where $\vec{v} = \frac{d\vec{r}}{dt}$ = velocity of particle at time t.

v_x, v_y, v_z = Cartesian components of velocity

Thus the time derivative of displacement vector gives velocity. Again differentiating eq. (5) with respect to t, we get the acceleration of particle defined as time derivative of velocity. Hence, the acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} \quad ...(6)$$

Since $\vec{v}$ itself is a derivative given as $\frac{d\vec{r}}{dt}$, we write

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$$

Eq. (6) may then be written in another way as

$$\vec{a} = \frac{d\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k} \quad ...(7)$$

Similarly, higher order derivatives of vectors are described.

Q.2. What do you mean by gradient of a scalar point function? Discuss its geometrical interpretation. Obtain expression for gradient in cartesian coordinate system.

Ans.

Gradient of Scalar Field : Directional Derivative

Consider a scalar field in which $\phi\ (x, y, z)$ is a scalar point function *i.e.*, a scalar quantity whose value depends on the values of position coordinates (x, y, z). If we change the position to $(x+dx, y+dy, z+dz)$ by altering three variables by infinitesimal amounts dx, dy and dz, the corresponding change in value of ϕ can be obtained using theorem of partial derivatives as

$$d\phi = \left(\frac{\partial\phi}{\partial x}\right)dx + \left(\frac{\partial\phi}{\partial y}\right)dy + \left(\frac{\partial\phi}{\partial z}\right)dz \quad ...(1)$$

where $d\phi$ = total change in the value of ϕ from position (x, y, z) to $(x+dx, y+dy, z+dz)$

Position vector of point (x, y, z) is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad ...(2)$$

Position vector of point $(x+dx, y+dy, z+dz)$ is

$$\vec{r}' = \vec{r} + d\vec{r} = (x+dx)\hat{i} + (y+dy)\hat{j} + (z+dz)\hat{k}$$

Thus, change in position vector

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \quad ...(3)$$

Now equation (1) can be written in the form of scalar product as

$$d\phi = \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

or $$d\phi = (\vec{\nabla}\phi)\cdot\vec{dr} \qquad ...(4)$$

where, $$\vec{\nabla} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \qquad ...(5)$$

$\vec{\nabla}$ is called del operator. The quantity $\vec{\nabla}\phi$ is called **gradient** of ϕ. We write :

$$\textbf{grad}\,\phi = \vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} \qquad ...(6)$$

Gradient of scalar function ϕ is a vector quantity.

Geometrical Interpretation of Gradient

Consider a scalar field with scalar point function $\phi(x, y, z)$. How fast does ϕ vary in space? The question has infinite answers, one for each direction we might choose to explore.

Let us consider two close surfaces S_1 and S_2 in the field such that (Fig.).

On S_1, $\phi(x, y, z) = \phi = C_1$, (a constant)

On S_2, $\phi(x, y, z) = \phi + d\phi = C_2$, (another constant)

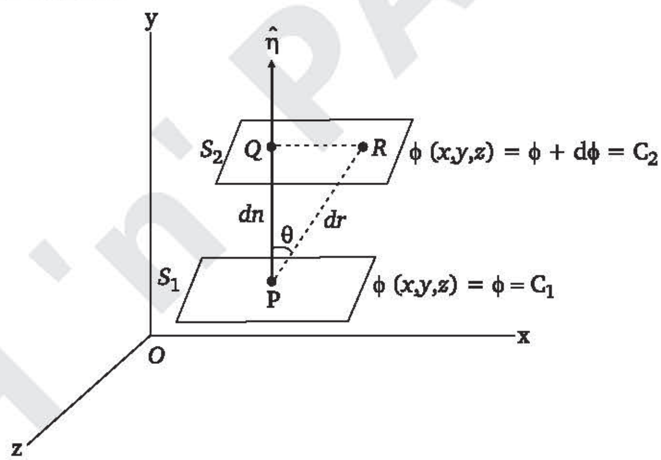


Fig.

From point P on surface S_1 to any point on surface S_2, increase in value of scalar function is same *i.e.*, $d\phi$, however space rate of increase is different. If we choose an arbitrary point R on surface S_2, such that distance $PR = dr$, then in the direction P to R

$$d\phi = \frac{\partial\phi}{\partial r}dr \qquad ...(7)$$

From different points on S_2, L.H.S. of eq. (1) is constant.

Let Q is a point on surface S_2 such that $PQ = dn$ is normal to surfaces S_1 at point. $P\cdot\vec{\eta}$ is unit vector along PQ. Since the least distance between surfaces is PQ, hence the max rate of increase of ϕ is $\frac{\partial\phi}{\partial\eta}$ *i.e.*,

$$(dr)_{\min} = dn \quad \text{and} \quad \left(\frac{\partial\phi}{\partial r}\right)_{\max} = \frac{\partial\phi}{\partial n} \qquad ...(8)$$

Hence from eq. (1)

$$d\phi = \left(\frac{\partial\phi}{\partial r}\right)_{\max}(dr)_{\min}$$

$\Rightarrow$ $$d\phi = \left(\frac{\partial \phi}{\partial \eta}\right) d\eta \quad ...(9)$$

From fig. $d\eta = dr \cos\theta$

$\therefore$ $$d\phi = \left(\frac{\partial \phi}{\partial \eta}\right) dr \cos\theta$$

$\therefore$ $$d\phi = \left(\frac{\partial \phi}{\partial \eta}\right) \hat{\eta} \cdot d\vec{r} \quad ...(10)$$

From eq. (4) $$d\phi = (\vec{\nabla}\phi) \cdot d\vec{r} \quad ...(11)$$

Comparing eq. (10) and (11)

$$\text{grad}\,\phi = \vec{\nabla}\phi = \left(\frac{\partial \phi}{\partial \eta}\right) \hat{\eta} \quad ...(12)$$

We may conclude : Gradient of a scalar field is a vector having magnitude equal to max rate of change of scalar function. It points in the direction of maximum rate of increase of function.

Examples :

1. In a conservative field, force $(\vec{F})$ is represented as gradient of potential energy (U)

$$\vec{F} = -\vec{\nabla} U$$

equation holds for both gravitational and electrostate field.

2. Electrostatic field $\vec{E}$ is represented as gradient of electrostatic potential (V) *i.e.*,

$$\vec{E} = -\vec{\nabla} V$$

Q.3. What do you mean by divergence of vector field? Discuss its geometrical interpretation and physical significance.

Ans.

Divergence of a Vector Field

Consider a vector field in which vector $\vec{A}$ is a continuously differentiable vector point function expressed as

$$\vec{A}(x,y,z) = A_x(x,y,z)\hat{i} + A_y(x,y,z)\hat{j} + A_z(x,y,z)\hat{k}$$

Divergence of $\vec{A}$ is defined as

$$\text{div}\,\vec{A} = \vec{\nabla} \cdot \vec{A}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})$$

i.e., $$\text{div}\,\vec{A} = \vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad ...(1)$$

From equation (1), it is clear that divergence of a vector function $\vec{A}$ is itself a scalar $\vec{\nabla} \cdot \vec{A}$.

Geometrical Interpretation

As the name divergence implies, $\vec{\nabla} \cdot \vec{A}$ is a measure of how much the vector $\vec{A}$ diverges (spreads out) from given point. For example, in fig. 1.

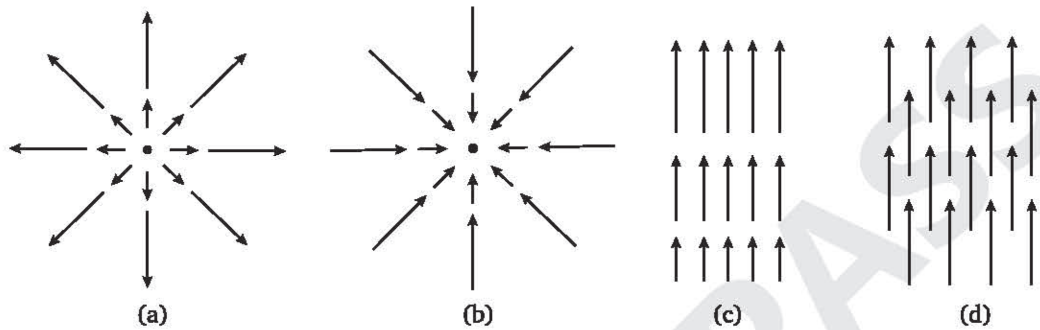


Fig. 1

(a) Fig. 1 (a) has large positive divergence.
(b) Fig. 1 (b) has large negative divergence.
(c) Fig. 1 (c) has positive divergence.
(d) Fig. 1 (d) has zero divergence.

Physical Significance

To develop the feeling of physical significance let $\vec{v}$ is a vector point function representing velocity of an incompressible fluid at point *P*. Consider an infinitesimal volume in the form of a parallelopiped of sides $\Delta x, \Delta y$ and Δz parallel to x, y, z axis respectively, *P* being the centre of parallelopiped (Fig. 2).

Let v_x, v_y and v_z are cartesian components of $\vec{v}$ at point *P*. Hence the component of velocity along *y*-axis at centre of face *OABC* is

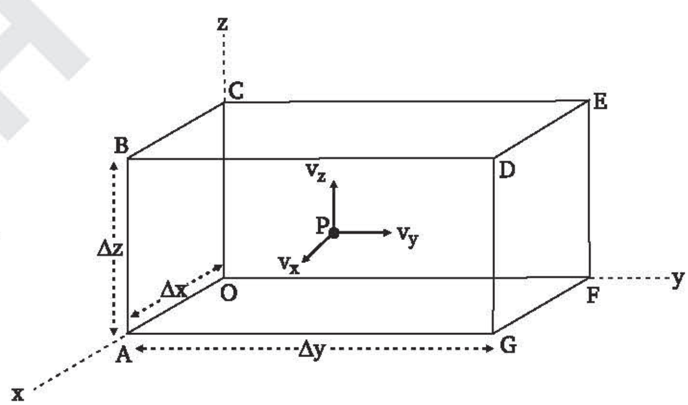


Fig. 2

$$v_y - \frac{1}{2} \cdot \frac{\partial v_y}{\partial y} \cdot \Delta y$$

Similarly the component of velocity along *y* axis at centre of face *DEFG* is

$$v_y + \frac{1}{2} \cdot \frac{\partial v_y}{\partial y} \cdot \Delta y$$

Since face area is infinitesimally small, above values may be taken as the values all over the face. Now the flux of liquid (*i.e.*, volume of liquid flowing per second) entering the face *OABC*.

$$= \text{Component of velocity normal to the face} \times \text{area of face}$$

$$= \left(v_y - \frac{1}{2} \cdot \frac{\partial v_y}{\partial y} \Delta y \right) \Delta x \, \Delta z$$

Similarly, the flux of liquid coming out of face *DEFG*

$$= \left(v_y + \frac{1}{2} \cdot \frac{\partial v_y}{\partial y} \Delta y \right) \Delta x \, \Delta z$$

Thus net outward flux from parallelopiped along y direction is

$$\left(v_y + \frac{1}{2} \cdot \frac{\partial v_y}{\partial y} \Delta y \right) \Delta x \, \Delta z - \left(v_y - \frac{1}{2} \cdot \frac{\partial v_y}{\partial y} \Delta y \right) \Delta x \, \Delta z = \frac{\partial v_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

Similarly, the net outward flux from parallelopiped along x direction

$$= \left(\frac{\partial v_x}{\partial x} \right) \Delta x \, \Delta y \, \Delta z$$

and, the net outward flux from parallelopiped along z direction

$$= \left(\frac{\partial v_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z$$

Thus, the total net outward flux from parallelopiped

$$= \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z$$

but $\quad \Delta x \, \Delta y \, \Delta z = \text{volume of parallelopiped.}$

Hence, the total net outward flux per unit volume from parallelopiped

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \vec{\nabla} \cdot \vec{v} = \text{div } \vec{v}$$

Thus, the divergence of a vector point function at a point represents net outward flux per unit volume around that point.

Solenoid Vector Function : A vector function $\vec{A}$ is said to be solenoidal if div $\vec{A} = \vec{\nabla} \cdot \vec{A} = 0$.

Q.4. State and prove Gauss-divergence theorem.

Ans.

Gauss-Divergence Theorem

This theorem presents a useful relation between a surface integral of a vector and the volume integral of the divergence of the vector.

This theorem states that :

"The surface integral of a vector $(\vec{A})$ over a closed surface S equals the volume integral of the divergence of vector over the volume V enclosed by the surface *i.e.*,"

$$\iint_S \vec{A} \cdot \vec{dS} = \iiint_V (\vec{\nabla} \cdot \vec{A}) \, dV$$

Thus, the theorem converts a volume integral into surface integral and vice versa.

Proof : Consider a closed surface S enclosing volume V in vector field $\vec{A}$. Let the entire volume be divided into infinitesimal volume elements in the form of parallelopiped of volume dV. Consider one such element shown in fig. with edges parallel to cartesian coordinate axes and their lengths dx, dy and dz along x, y, z axis respectively.

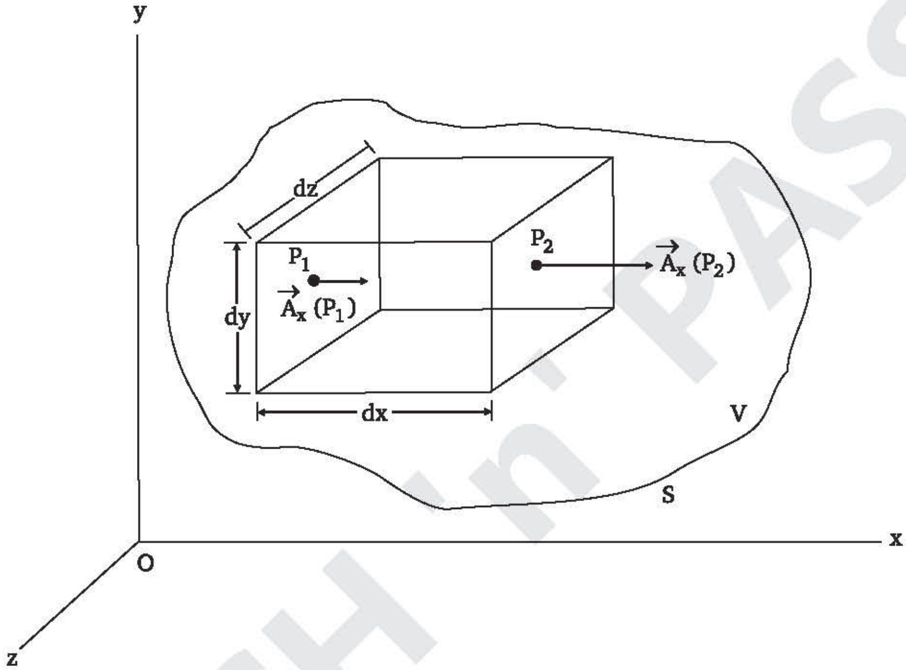


Fig.

Let $P_1(x, y, z)$ and $P_2(x, dx, y, z)$ are centres of left and right faces of parallelopiped which lie in $Y-Z$ plane.

Since the sides of parallelopiped are infinitesimal, value of X component of $\vec{A}$ at entire face may be taken as its value at centre of face.

The flux outward through left face $= -A_x(P_1)\,dydz$.

The flux outward through right face $= A_x(P_2)\,dydz$.

Thus, total outward flux through volume element x-direction

$$\{A_x(P_2) - A_x(P_1)\}\,dydz$$

$$= \frac{\partial A_x}{\partial x} \cdot dxdydz \qquad ...(1)$$

Similarly, total outward flux through volume element in Y-direction

$$= \frac{\partial A_y}{\partial y}\,dxdydz$$

and total outward flux through volume element in z direction.

$$= \frac{\partial A_z}{\partial y} dxdydz$$

Total outward flux through all faces of volume element

$$\iint_{\text{Surface of parallelopiped}} \vec{A} \cdot \vec{dS} = \left(\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial y} \frac{\partial A_z}{\partial z} \right) dxdydz$$

$$= (\vec{\nabla} \cdot \vec{A}) dV$$

Summing up the outward fluxed through all such elements into which entire volume is divided (As $dV \to 0 \Rightarrow \vec{dS} \to 0 \Rightarrow$ summation $\to$ integration).

$$\iint_S \vec{A} \cdot \vec{dS} = \iiint_V (\vec{\nabla} \cdot \vec{A}) dV$$

This is Gauss-Divergence Theorem.

Q.5. If $\vec{A} = 5t^2 \hat{i} + t \hat{j} - t^3 \hat{k}$ and $\vec{B} = (\sin t)\hat{i} + (\cos t)\hat{j}$. Find

(a) $\frac{d}{dt}(\vec{A} \cdot \vec{B})$ **(b) $\frac{d}{dt}(\vec{A} \times \vec{B})$**

Sol. (a) $\vec{A} \cdot \vec{B} = 5t^2 \sin t - t \cos t$

$$\frac{d}{dt}(\vec{A} \cdot \vec{B}) = 5[t^2 \cos t + 2t \sin t] - [t(-\sin t) + \cos t]$$

$$= 5t^2 \cos t + 10t \sin t + t \sin t - \cos t$$

$$= (5t^2 - 1)\cos t + 11t \sin t$$

(b)

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$= \hat{i}(0 - t^3 \cos t) + \hat{j}(-t^3 \sin t - 0) + \hat{k}(-5t^2 \cos t - t \sin t)$$

$$= -(t^3 \cos t)\hat{i} - (t^3 \sin t)\hat{j} - (5t^2 \cos t + t \sin t)\hat{k}$$

$\therefore$

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = -[t^3(-\sin t) + 3t^2 \cos t]\hat{i} - [t^3(\cos t + 3t^2 \sin t]\hat{j}$$

$$-[5t^2(-\sin t) + 10t \cos t + t \cos t + \sin t]\hat{k}$$

$$= (t^3 \sin t - 3t^2 \cos)\hat{i} - (t^3 \cos t + 3t^2 \sin t)\hat{j}$$

$$+ (5t^2 \sin t + t \cos t + \sin t)\hat{k}$$

Q.6. State and prove Stoke-Curl Theorem.

Ans.

Stoke-Curl Theorem

This theorem relates a surface integral over an open surface to the line integral around the curve binding the surface.

This theorem states that

"Surface integral of curl of a vector $\vec{A}$ over an open surface S is equal to line integral of vector $\vec{A}$ along around the curve C binding the surface as perimeter *i.e.*,"

$$\iint_S \text{curl}\,\vec{A}\cdot\vec{dS} = \oint_C \vec{A}\cdot\vec{dl} \quad \text{or} \quad \iint_S (\vec{\nabla}\times\vec{A})\cdot\vec{dS} = \oint_C \vec{A}\cdot\vec{dl} \qquad ...(1)$$

Thus stoke theorem converts a surface integral into line integral and vice versa.

Proof : Let us consider a plane surface S of arbitrary shape bounded by close curve C inside a vector field $\vec{A}$ (fig). Let the surface be divided into large no. area elements by the network of curves.

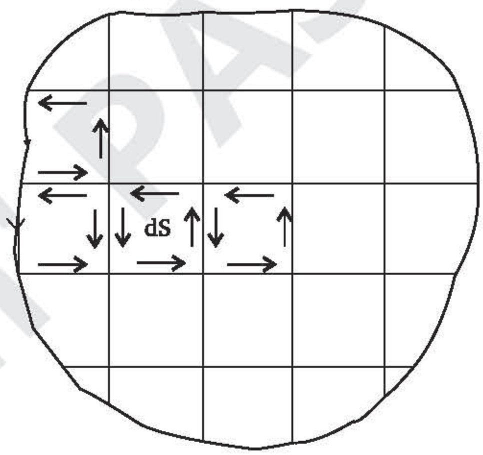


Fig.

Consider on such element enclosing area dS. By definition of curl, curl of $\vec{A}$ at centre of this element is given as

$$\text{curl}\,\vec{A} = \frac{1}{dS}\oint_{c'} \vec{A}\cdot\vec{dl}$$

where $\oint_{c'} \vec{A}\cdot\vec{dl}$ =line integral of $\vec{A}$ along the perimeter c' of this element.

Conversely $\qquad \oint_{c'} \vec{A}\cdot\vec{dl} = \text{curl}\,\vec{A}\cdot\vec{dl} \qquad ...(2)$

Now, if we sum up left hand side *i.e.*, the circulations around all these small loops binding the area elements, we find that the contributions due to all binding lines of loops, except those forming part of curve C itself, cancel out since these contributions are in the form of equal and oppositely directed pairs. Thus the summation simply gives circulation of line integral of vector $\vec{A}$ around original closed curve C. *i.e.*,

$$\Sigma\oint_{c'} \vec{A}\cdot\vec{dl} = \oint_C \vec{A}\cdot dl \qquad ...(3)$$

Similarly, $\qquad \Sigma\,\text{curl}\,\vec{A}\cdot\vec{dl} = \iint_S \text{curl}\,\vec{A}\cdot\vec{dS} \qquad ...(4)$

Since no of element are infinitely large summation $\rightarrow$ integration.

Hence, applying summation on eq. (2) and putting values from equations (3) and (4).

$$\oint_C \vec{A}\cdot\vec{dl} = \iint_S \text{curl}\,\vec{A}\cdot\vec{dS}$$

This is **Stoke** theorem.

Q.7. For a position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find the values of :

(i) grad $\left(\frac{1}{r}\right)$ (ii) grad r^n (iii) div $(r^n \vec{r})$ (iv) curl $\left(\frac{\vec{r}}{r^3}\right)$

Sol. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, therefore $r = (x^2 + y^2 + z^2)^{1/2}$

(i) $$\text{grad}\left(\frac{1}{r}\right) = \vec{\nabla}\left(\frac{1}{r}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} + \hat{j}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + \hat{k}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial z}$$

Here, $$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{1/2} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2x = \frac{x}{r}$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$\therefore$ $$\text{grad}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\left[\hat{i}\cdot\frac{x}{r} + \hat{j}\cdot\frac{y}{r} + \hat{k}\cdot\frac{z}{r}\right] = -\frac{\vec{r}}{r^3}$$

(ii) $$\text{grad } r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(r^n)$$

$$= nr^{n-1}\left(\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right)$$

$$= nr^{n-1}\left[\hat{i}\left(\frac{x}{r}\right) + \hat{j}\left(\frac{y}{r}\right) + \hat{k}\left(\frac{z}{r}\right)\right] = nr^{n-2}\vec{r}$$

(iii) $$\text{div}(r^n \vec{r}) = \phi\vec{\nabla}\cdot(r^n \vec{r})$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= \left(r^n + x\cdot nr^{n-1}\frac{\partial r}{\partial x}\right) + \left(r^n + y\cdot nr^{n-1}\frac{\partial r}{\partial y}\right) + \left(r^n + x\cdot nr^{n-1}\frac{\partial r}{\partial z}\right)$$

$$= 3r^n + nr^{n-1}\left[x + \frac{\partial r}{\partial x} + y + \frac{\partial r}{\partial y} + z + \frac{\partial r}{\partial z}\right]$$

$$= 3r^n + nr^{n-1}\left[x\left(\frac{x}{r}\right) + y\left(\frac{y}{r}\right) + z\left(\frac{z}{r}\right)\right]$$

$$= 3r^n + nr^{n-1}\left[\frac{r^2}{r}\right] = (3+n)r^n$$

(iv)

$$\frac{\vec{r}}{r^3} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3}$$

$\therefore$

$$\text{Curl}\left(\frac{\vec{r}}{r^3}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^3}\right)\right] + \hat{j}\left[\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) - \frac{\partial}{\partial x}\left(\frac{z}{r^3}\right)\right] + \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{y}{r^3}\right) - \frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right]$$

$$= \hat{i}\left[\frac{-3z}{r^4}\cdot\frac{\partial r}{\partial y} + \frac{3y}{r^4}\cdot\frac{\partial r}{\partial z}\right] + \hat{j}\left[\frac{-3x}{r^4}\cdot\frac{\partial r}{\partial z} + \frac{3z}{r^4}\cdot\frac{\partial r}{\partial x}\right] + \hat{k}\left[\frac{-3y}{r^4}\cdot\frac{\partial r}{\partial x} + \frac{3x}{r^4}\cdot\frac{\partial r}{\partial y}\right]$$

$$= \frac{3}{r^4}\left[-z\left(\frac{y}{r}\right) + y\left(\frac{z}{r}\right)i\right] + \frac{3}{r^4}\left[-x\left(\frac{z}{r}\right) + z\left(\frac{x}{r}\right)j\right] + \frac{3}{r^4}\left[-y\left(\frac{x}{r}\right) + x\left(\frac{y}{r}\right)\hat{k}\right]$$

$$= \vec{0}$$

We can say $\frac{\vec{r}}{r^3}$ is irrotational.

Q.8. Define the dirac delta function and discuss it important properties.

Ans.

Dirac Delta Function

The Dirac delta function is a function introduced in 1930 by **P.A.M. Dirac**. It is the name given to a mathematical structure that is intended to represent idealized point object (such as point mass or point charge) or to describe impulsive force. It is most widely used in quantum mechanics.

Physical Model : Consider a physical model that visualizes a delta function distribution of unit mass on x-axis in small region at origin.

Let $\mu(x) = $ linear density = mass per unit length

$$\mu(x) = 0 \text{ if } x < -\varepsilon$$
$$\mu(x) = 0 \text{ if } x > -\varepsilon$$
$$\mu(x) = \text{finite } -\varepsilon < x < \varepsilon$$

where ε small. $\mu(x)$ *vs* x graph is shown in fig 1 (a).

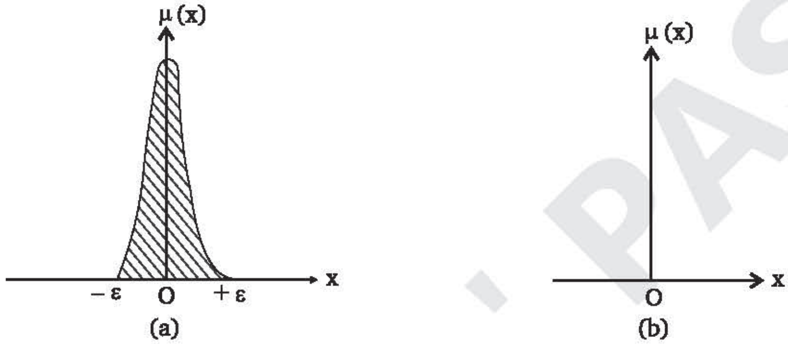


Fig. 1

Area under the curve

$$\int_{-\infty}^{\infty} \mu(x)\, dx = \text{total mass} = 1$$

If $\quad \varepsilon \to 0$, distribution $\to$ point mass

$\mu(x)$ *vs* x curve will become as shown in figure 1 (b).

The curves showing linear charge density *vs* x graph due to a point charge at origin and impulsive force *vs* time can be interpretend in same way.

Definition of Dirac Delta Function

The Dirac delta function is represented with the Greek lowercase symbol delta, written as function; $\delta(x)$. It is defined by the following property :

$$\delta(x) = \begin{Bmatrix} 0 & \text{if} & x \neq 0 \\ \infty & \text{if} & x = 0 \end{Bmatrix} \qquad ...(1)$$

with $\quad \displaystyle\int_{-\infty}^{\infty} \delta(x)\, dx = 1 \qquad ...(2)$

This can be pictured as infinitely high, infinitely narrow spike at $x = 0$, shown in fig 2 (a) which is limiting case of figure 2 (b).

L.H.S. of equation (2) represents area under the curve which might by unity.

Mathematically delta function is not a function at all since it is too singular and its value is not finite at $x = 0$. It is a generalized idea of functions which can be used inside integrals. In mathematical literature it is called **generalized function** or **distribution**.

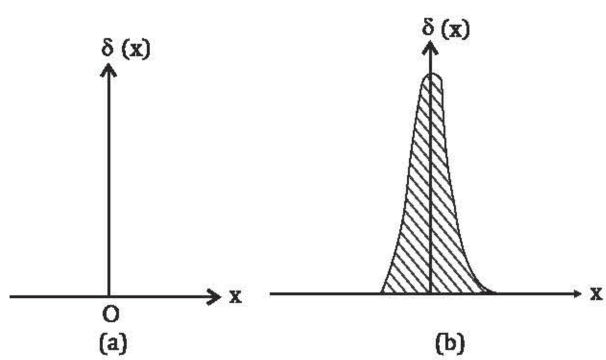


Fig. 2

Important Properties

1. $\delta(-x)=\delta(x)$ *i.e.*, $\delta(x)$ is even function.

Proof : We know $\int_{-\infty}^{\infty}\delta(x)\,dx=1$

if $x\to -x$, limits change ∞ to -8

$$\Rightarrow \qquad \int_{\infty}^{-\infty}\delta(-x)\,d(-x)=1$$

$$\Rightarrow \qquad -\int_{\infty}^{-\infty}\delta(-x)\,dx=1$$

$$\Rightarrow \qquad \int_{\infty}^{-\infty}\delta(-x)\,dx=1$$

From above

$$\Rightarrow \qquad \int_{-\infty}^{\infty}\delta(x)\,dx=\int_{-\infty}^{\infty}\delta(-x)\,dx$$

i.e., $$\delta(x)=\delta(-x) \qquad ...(3)$$

2. $$x\delta(x)=0$$

Proof : If $x=0, \delta(x)\neq 0$ and if $x\neq 0, \delta(x)=0$, therefore all values of x.

$$x\,\delta(x)=0 \qquad ...(4)$$

3. If $f(x)$ is continuous function

$$f(x)\delta(x)=f(0)\,\delta x$$

Proof : $\delta(X)=0$ everywhere except at $x=0$. It means the product $f(x)\delta(x)=0$ everywhere except at $x=0$. It follows that

$$f(x)\delta(x)=f(0)\,\delta(x) \qquad ...(5)$$

4. $\int_{-\infty}^{\infty}f(x)\delta(x)\,dx=f(0)$

Proof :

$$\text{L.H.S.}=\int_{-\infty}^{\infty}f(x)\delta(X)\,dx$$

$$=\int_{-\infty}^{\infty}f(0)\delta(x)\,dx \qquad \text{[using property (3)]}$$

$$=f_0\int_{-\infty}^{\infty}\delta(x)\,dx$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty}f(x)\delta(x)\,dx=f(0) \qquad ...(6)$$

In equation (4), $\delta(x)$ can be regarded as operator. Which pulls the value of function at $x=0$.

5. $\delta(ax)=\frac{1}{a}\delta(x)$, a is a constant, $a>0$

Proof : Multiply L.H.S. by $f(x)$ and integrate over $-\infty$ to ∞.

$$\int_{-\infty}^{\infty}f(x)\,\delta(ax)\,dx=I \text{ (Let)}$$

Substitute $$ax=y \Rightarrow x=\frac{y}{a} \text{ and } dx=\frac{dy}{a}$$

$\therefore$ $$I = \int_{-\infty}^{\infty} f\left(\frac{y}{a}\right)\delta(y)\frac{dy}{a} = \frac{1}{a}\int_{-\infty}^{\infty} f\left(\frac{y}{a}\right)\delta(y)\,dy$$

$$\frac{1}{a}f(0) = \frac{1}{a}\int_{-\infty}^{\infty} f(x)\delta(x)\,dx$$

Thus, $$\int -\infty^{\infty} f(x)\delta(ax)\,dx = \frac{1}{a}\int_{-\infty}^{\infty} f(x)\delta(x)\,dx$$

i.e., $$\delta(ax) = \frac{1}{a}\delta(x) \qquad ...(7)$$

6. Shifting Property : If we shift the spike of $\delta(x)$ *vs* x curve from $x = 0$ to $x = a$, as shown in fig (3). Then as equation (1) and (2) become

$$\delta(x-a) = \begin{cases} 0 & \text{if} \quad x \neq 0 \\ \infty & \text{if} \quad x = 0 \end{cases} \qquad ...(8)$$

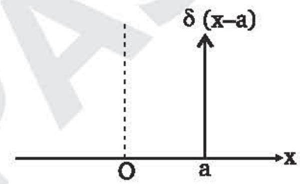


Fig. 3

and $$\int_{-\infty}^{\infty} \delta(x-a)\,dx = 1 \qquad ...(9)$$

Then eq. (5) and (6) generalize to

$$f(x)\delta(x-a) = f(a)\delta(x-a) \qquad ...(10)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a) \qquad ...(11)$$

7. Differentiation properties :

(a) $$\delta'(-x) = -\delta'(x)$$

Proof : $$\text{L.H.S.} = \frac{\partial}{\partial(-x)}\delta(-x) = -\frac{\partial}{\partial x}\delta(-x)$$

$$= -\frac{\partial}{\partial(x)}\delta(x) \qquad [\because \delta(-x) = \delta(x)]$$

$\Rightarrow$ $$\delta'(-x) = -\delta'(x) \qquad ...(12)$$

(b) $$\boldsymbol{x\frac{d}{dx}\delta(x) = -\delta(x)}$$

Proof : $$\int_{-\infty}^{\infty} x\frac{d}{dx}\delta(x)\,dx = x\,\delta(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\delta(x)\,dx \qquad \text{(Integration by parts)}$$

$$= 0 - \int_{-\infty}^{\infty}\delta(x)\,dx \qquad [\because x\,\delta(x) = 0]$$

$\Rightarrow$ $$\int_{-\infty}^{\infty} x\frac{d}{dx}\delta(x)\,dx = -\int_{\infty}^{\infty}\delta(x)\,dx$$

Comparing both sides $$x\frac{d}{dx}\delta(x) = -\delta(x) \qquad ...(13)$$

❑

UNIT-III

Coordinate Systems

SECTION-A VERY SHORT ANSWER TYPE QUESTIONS

Q.1. What is meant by a coordinate system?

Ans. Coordinate systems are used to describe the position of an object in space. A coordinate system is essentially an artificial mathematic tool to describe the position of a real object. It also helps us to understand the motion of particle or a system of particles.

Q.2. What is the meaning by pseudo force?

Ans. A physically apparent but non existent force felt by an observor in a non-inertial frame (that is, a frame undergoing acceleration). Newton's law of motion hold true mithin such as reference frame only it the existence of such a force is presumed. The centrifugal force is an example of a pseudo force.

Q.3. What is centrifugal force of Earth?

Ans. Centrifugal force is the apparent outward force on a mass when it is rotated. Since earth rotates around a fixed axis, the direction of centrifugal force is always outward away from the axis. Thus, it is opposite to the direction of gravity at the equator, at earth poles it is zero.

Q.4. Draw unit vector $\hat{\gamma}$ and $\hat{\theta}$ in polar coordinate system.

Ans. The radial unit vector $\hat{r}$ and tangential unit vector $\hat{\theta}$ are taken along the direction of increasing r and increasing θ respectively.

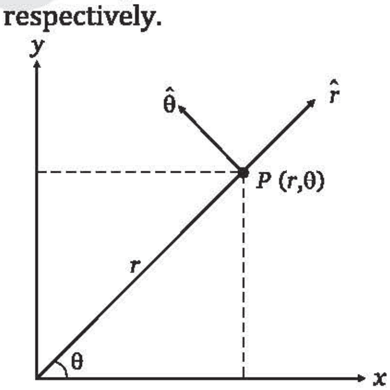


Fig.

Q.5. Drive the expression for the line and volume elements in cyclindrical coordinates.

Ans. Infinitesimal displacement in $\hat{r}$ direction $= dr$

Infinitesimal displacement in $\hat{\theta}$ direction $= rd\theta$

Infinitesimal displacement in $\hat{z}$ direction $= dz$.

Therefore, displacement vector from (r, θ, z) to $(r + dr, \theta + d\theta, z + dz)$

$$\vec{dl} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + dz\,\hat{k}$$

The volume element $dV = dr\, r\, d\theta\, dz$

$$dV = rdr\, d\theta\, dz$$

Q.6. Show that the plane polar coordinates are orthogonal.

Sol.

$$r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\hat{r}\cdot\hat{\theta} = -\sin\theta\cos\theta + \sin\theta\cos\theta = 0.$$

Q.7. What are polar coordinates used for?

Ans. Polar coordinates are used often in navigation as the destination or direction of travel can be given as an angle and distance from the object being considered. For instance, aircraft use a slightly modified version of the polar coordinates for navigation.

Q.8. The polar coordinates of a point are $r, \theta, \phi = 8, 30°$ and $45°$. Find the Cartesian coordinates of same point.

Sol.

$$x = r\sin\theta\cos\phi = 8\sin 30\cos 45 = 8\times\frac{1}{2}\times\frac{1}{\sqrt{2}} = 2\sqrt{2}$$

$$y = r\sin\theta\sin\phi = 8\sin 30\sin 45 = 8\times\frac{1}{2}\times\frac{1}{\sqrt{2}} = 2\sqrt{2}$$

$$z = r\cos\theta = 8\cos 30 = 8\times\frac{\sqrt{3}}{2} = 4\sqrt{3}$$

SECTION-B SHORT ANSWER TYPE QUESTIONS

Q.1. Differentiate between one dimensional and two dimensional coordinate systems.

Ans.

One Dimensional Coordinate Systems

The most easiest coordinate system to describe the location of objects is one dimensional $(1 - D)$ space. In this coordinate system only one coordinate is needed to define the location of an object. Suppose a train is at position $x = 0$ (x is a single real number), which we can call as origin. If it is constrained to move along a straight line (say x-axis), then all points on the east side of the origin corresponds to positive values of x and that of west side corresponds to the negative values of x.

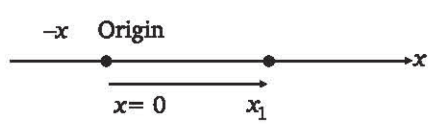


Fig. 1

Hence x-axis is our coordinate system in which we can represent the location of a body by drawing an arrow starting from the origin in the direction of increasing x.

Two Dimensional (2-*D*) Cartesian Coordinate Systems

If a particle moves in a plane but not along a straight line, then $1-D$ coordinate system is not appropriate to describe the position of the particle, but a $2-D$ Cartesian coordinate system is used. A $2-D$ Cartesian coordinate system is formed by two mutually perpendicular axes. The two axes intersect at the point O, called the origin. In the right handed system, one of the axes directed to the right is called x-axis, while the other directed vertically upwards is called y-axis. The coordinate of a point P on the $x-y$ plane is determined by two real numbers x and y. If we drop a perpendicular from point P on x-axis the foot of perpendicular meets on x-axis at M. The distance of M from the origin of the coordinate system is called x-coordinate of the point or abscissa and similarly the distance of foot of perpendicular dropped from P on y-axis from the origin is called y-coordinate of the point or ordinate *i.e.*,

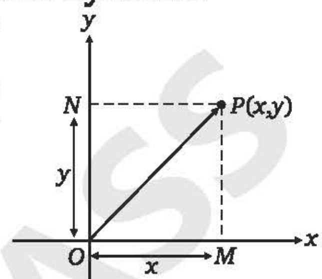


Fig. 2.

$$OM = x; \qquad ON = y$$

and point P is denoted as (x, y).

If $\hat{i}$ and $\hat{j}$ be the unit vectors along x and y-axes, then the position vector of point P with respect to the origin is given by

$$\vec{OP} = x\,\hat{i} + y\,\hat{j}$$

The unit vectors are taken along increasing directions of x and y respectively. The distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lying on Cartesian plane is

$$P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Q.2. Derive the expression of the basic vectors in cylindrical polar system.

Sol. Basic Vector in Cylindrical Polar Coordinates : The basic vectors in cylindrical coordinate system are $\hat{r}, \hat{\theta}$ and $\hat{k}$

$$\vec{r} = x\,\hat{i} + y\,\hat{j}$$

$$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$$

$$\therefore \qquad \hat{r} = \frac{\vec{r}}{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

Unit vector $\hat{\theta}$ is perpendicular to $\hat{r}$

$$\hat{\theta} = \cos\left(\frac{\pi}{2} + \theta\right)\hat{i} + \sin\left(\frac{\pi}{2} + \theta\right)\hat{j}$$

$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

Also

$$\hat{z} = \hat{k}$$

Also $\hat{r} \times \hat{\theta} = \hat{k};\ \hat{\theta} \times \hat{k} = \hat{r};\ \hat{k} \times \hat{r} = \hat{\theta}$

Displacement vector, arc length and area elements in plane polar coordinates : In plane polar coordinate system, two coordinates of a point are (r, θ). This point is displaced to $(r + dr, \theta + d\theta)$. Then

An infinitesimal element of length in r-direction $= dr$

and an infinitesimal element of length in $\hat{\theta}$-direction $= r\, d\theta$

Therefore the line element or infinitesimal displacement can be written as

$$\vec{dl} = dr\, \hat{r} + r\, d\theta\, \hat{\theta}$$

The area element can be written as

Fig.

$$\vec{dS} = dr\, \hat{r} \times r\, d\theta\, \hat{\theta}$$

$$= r\, dr\, d\theta\, (\hat{r} \times \hat{\theta}) = r\, dr\, d\theta\, \hat{n}$$

where $\hat{n}$ is a unit vector normal to $r - \theta$ planes containing area elements ds.

Q.3. Derive relationship beteen line, volume and surface elements in spherical polar coordinates.

Sol. Line, Volume and Surface Elements in Spherical Polar Coordinates : Let there be a point with coordinates (r, θ, ϕ). It is displaced to a new location $(r + dr, \theta + d\theta, \phi + d\phi)$.

An infinitesimal element of length in $\hat{r}$ direction $= dr$.

An infinitesimal element of length in $\hat{\theta}$ direction = length of an arc $= r\, d\theta$

An infinitesimal element of length in $\hat{\phi}$ direction $= r \sin\theta\, d\phi$

The displacement vector or line element $\vec{dl}$ can be written as

$$\vec{dl} = dr\, \hat{r} + r\, d\theta\, \hat{\theta} + r \sin\theta\, d\phi\, \hat{\phi}$$

The infinitesimal volume element

$$dV = dr\, r\, d\theta\, r \sin\theta\, d\phi$$

$$= r^2 \sin\theta\, dr\, d\theta\, d\phi$$

For surface element, there is no general expression as it depends on the orientation.

Let r remains constant and θ and ϕ are allowed to vary. Then area element

$$\vec{dS_1} = r\, d\theta\, \hat{\theta} \times r \sin\theta\, \hat{\phi}$$

$$= r^2 \sin\theta\, d\theta\, (\hat{\theta} \times \hat{\phi}) = r^2 \sin\theta\, d\theta\, \hat{r}$$

Similarly if θ remains constant and r and ϕ are allowed to vary. Then area element.

$$\vec{dS_2} = dr\, \hat{r} \times r \sin\theta\, d\phi\, \hat{\phi}$$

$$= r\, dr \sin\theta\, d\phi\, \hat{\theta}$$

Q.4. A particle masses in a plane with $\dot{r} = 4$ m/s and $\dot{\theta} = 2$ rad/s. Find the velocity and acceleration of the particle when $r = 3$ m.

Sol.

$$\vec{v} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}.$$

$$= 4\hat{r} + 3 \times 2\hat{\theta} = (4\hat{r} + 6\hat{\theta})\text{ m/sec.}$$

$$|\vec{v}| = \sqrt{4^2 + 6^2} = \sqrt{16 + 36} = \sqrt{52};\ \tan\phi = \frac{6}{4}.$$

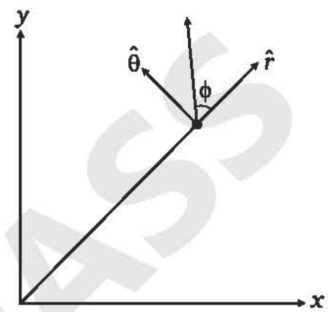


Fig.

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = 0; \quad \ddot{\theta} = \frac{d\dot{\theta}}{dt} = 0$$

$$\vec{a} = -r\dot{\theta}^2\,\hat{r} + 2\dot{r}\dot{\theta}\,\hat{\theta} = -3 \times 4\hat{r} + 2 \times 4 \times 2\hat{\theta}$$

$$= -12\hat{r} + 16\hat{\theta}.$$

Q.5. Derive the expression of the gradient, divergence and curl in spherical polar coordinate system.

Sol. **Gradient, Divergence and Curl in Spherical Polar Coordinate System :** In spherical polar coordinate system

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

Differentially all of these transformation equations :

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}d\phi$$

$$dx = \sin\theta\cos\phi\,dr + r\cos\theta\cos\phi\,d\theta - r\sin\theta\sin\phi\,d\phi$$

Similarly, $$dy = \sin\theta\sin\phi\,dr + r\cos\theta\sin\phi\,d\theta + r\sin\theta\cos\phi\,d\phi$$

$$dz = \cos\theta\,dr - r\sin\theta\,d\theta$$

Now the line element dS can be written as

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2$$

It can be seen from this equation that

$$q_1 = r, q_2 = \theta, q_3 = \phi$$
$$h_1 = 1, h_2 = r, h_3 = r\sin\theta$$
$$\hat{u}_1 = \hat{r}, \hat{u}_2 = \hat{\theta}, \hat{u}_3 = \hat{\phi}$$

$$\vec{\nabla} S = \frac{\hat{u}_1}{h_1}\frac{\partial S}{\partial q_1} + \frac{\hat{u}_2}{h_2}\frac{\partial S}{\partial q_2} + \frac{\hat{u}_3}{h_3}\frac{\partial S}{\partial q_3}$$

$$\vec{\nabla} S = \hat{r}\frac{\partial S}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial S}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial S}{\partial \phi}$$

or

$$\vec{\nabla} \to \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Now,

$$\vec{\nabla}\cdot\vec{\nabla} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial q_1}(V_1h_2h_3) + \frac{\partial}{\partial q_2}(V_2h_3h_1) + \frac{\partial}{\partial q_3}(V_3h_1h_2)\right\}$$

$$\vec{\nabla}\cdot\vec{\nabla} = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}(r^2V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$$

(Here $\vec{V} = V_r\hat{r} + V_\theta\hat{\theta} + V_\phi\hat{\phi}$)

$$\vec{\nabla}\times\vec{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ V_r & rV_\theta & r\sin\theta\, V_\phi \end{vmatrix}$$

Q.6. A bucket of water *u* allowed to spin with angular speed ω. What will be the shape of the water surface?

Sol. Consider a small *m* of water at the surface of the liquid. The force acting on it are :

(i) Normal force *N* (ii) Weight (iii) Pseudo force

$$N\cos\theta = mg$$

$$N\sin\theta = mr\omega^2$$

$$\tan\theta = \frac{\omega^2 r}{g}$$

Also,

$$\tan\theta = \frac{dz}{dr} = \frac{\omega^2 r}{g}$$

$$dz = \frac{\omega^2 r\, dr}{g}$$

$$z = \frac{L}{2}\frac{\omega^2 r^2}{g}.$$

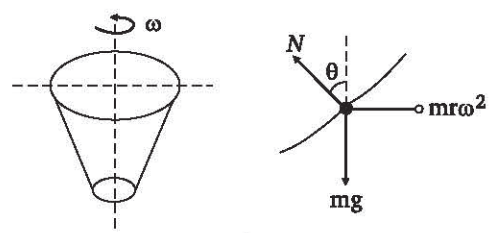


Fig.

Therefore, the surface is the paraboloid of revolution.

Q.7. Write short note on velocity and accelaration in cylindrical coordinates.

Ans. **Velocity and Accelaration in Cylindrical Co-ordinates**

The analysis of polar coordinates can be extended to three dimensions cylindrical coordinates by adding *z*-coordinate. Every point in space is determined by *r* and θ coordinates of its projection in $x - y$ plane and it *z* coordinate.

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

Also
$$\frac{d\hat{r}}{dt} = \dot{\theta}\,\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\,\hat{r}\,; \qquad \frac{d\hat{k}}{dt} = 0$$

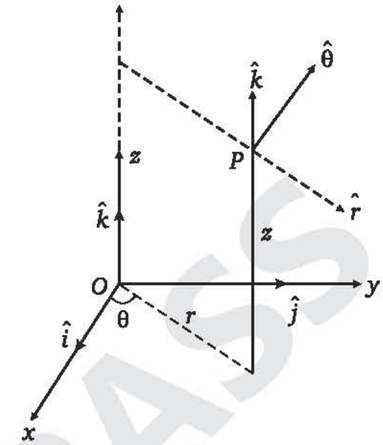


Fig.

The position vector of P can be written as

$$\vec{r} = r\,\hat{r} + z\,\hat{k} \text{ and velocity } \vec{u} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta} + \dot{z}\,\hat{k}$$

Finally the acceleration can be written as

$$\vec{a} = (\ddot{r} - r\,\dot{\theta}^2)\,\hat{r} + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})\,\hat{\theta} + \ddot{z}\,\hat{k} = a_r\,\hat{r} + a_\theta\,\hat{\theta} + a_z\,\hat{k}$$

where $\quad a_r = \ddot{r} - r\,\dot{\theta}^2 ; a_\theta = r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} ; a_z = \ddot{z} \quad$ and $\quad a = \sqrt{a_r^2 + a_\theta^2 + a_z^2}$

Also the equation of motion in component form can be written as

$$F_r = ma_r = m(\ddot{r} - r\,\dot{\theta}^2)$$

$$F_\theta = ma_\theta = m(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})$$

$$F_z = ma_z = m\ddot{z}$$

Q.8. Explain the basic vectors in spherical coordinate system.

Ans.

Basic Vectors in Spherical Coordinate System

The basic vectors or unit vectors in a spherical coordinate system are $\hat{r}, \hat{\theta}$ and $\hat{\phi}$ which are directed respectively along the direction of increasing r, θ and ϕ. The surface of constant r are sphere of radius r.

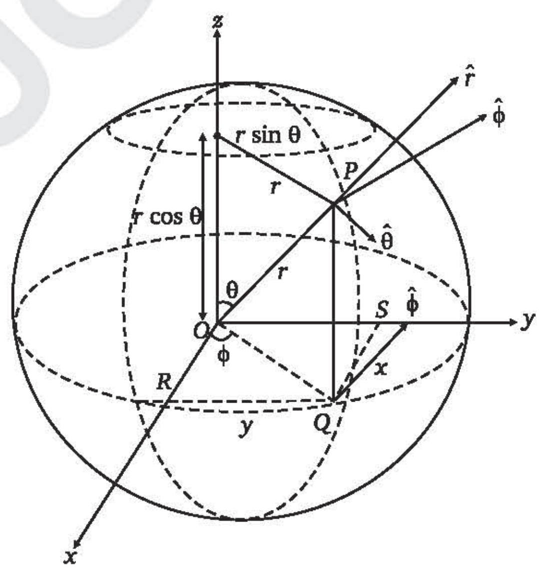


Fig.

The surface of constant θ is a cone of semi angle θ about the z-axis. A surface of constant ϕ is a plane containing the z-axis which makes an angle ϕ with the reference plane :

$$\vec{r} = \vec{OP} = \vec{OQ} + \vec{QP} = r \sin\theta\, \hat{r} + r\cos\theta\, \hat{k}$$

Also $$\hat{r} = \cos\theta\, \hat{i} + \sin\phi\, \hat{j}$$

$$\vec{r} = r\sin\theta\cos\phi\, \hat{i} + r\sin\theta\sin\phi\, \hat{j} + r\cos\theta\, \hat{k}$$

$$\hat{r} = \sin\theta\cos\phi\, \hat{i} + \sin\theta\sin\phi\, \hat{j} + \cos\theta\, \hat{k}$$

The unit vector $\hat{\theta}$ is perpendicular to $\hat{r}$

$\therefore$ $$\hat{\theta} = \sin\left(\frac{\pi}{2}+\theta\right)\cos\phi\, \hat{i} + \sin\left(\frac{\pi}{2}+\theta\right)\sin\phi\, \hat{j} + \cos\left(\frac{\pi}{2}+\theta\right)\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\, \hat{i} + \cos\theta\sin\phi\, \hat{j} - \sin\theta\, \hat{k}$$

To find out $\hat{\phi}$, let PQ be perpendicular to xy plane

$$OQ = r\sin\theta = \rho \text{ (say)}$$

The unit vector $\hat{\phi}$ will be perpendicular to $\hat{\rho}$.

$$\vec{\rho} = \rho\cos\phi\, \hat{i} + \rho\sin\phi\, \hat{j}$$

$$\hat{\rho} = \cos\phi\, \hat{i} + \sin\phi\, \hat{j}$$

$\therefore$ $$\hat{\phi} = \cos\left(\frac{\pi}{2}+\phi\right)\hat{i} + \sin\left(\frac{\pi}{2}+\phi\right)\hat{j}$$

$$\hat{\phi} = -\sin\phi\, \hat{i} + \cos\phi\, \hat{j}$$

The unit vectors $\hat{r}, \hat{\theta}$ and $\hat{\phi}$ are orthogonal vectors and follows :

$$\hat{r} \times \hat{\theta} = \hat{\phi}$$

$$\hat{\theta} \times \hat{\phi} = \hat{r}$$

$$\hat{\phi} \times \hat{r} = \hat{\theta}$$

Q.9. Show that the motion of one projectile as seen from another projectile will always be a straight line motion.

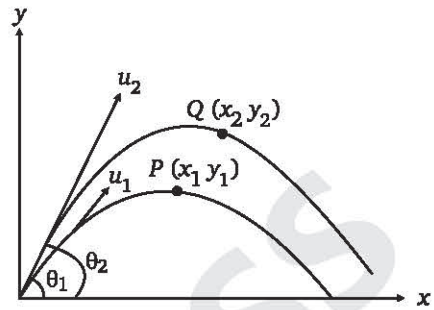
Sol. Let us consider two projectiles with their initial velocity are angles of projections as $u_1, u_2, \theta_1, \theta_2$ respectively.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on two projectile paths at a time t.

$$x_1 = u\cos\theta_1 t$$

$$y_1 = u_1 \sin\theta_1 t - \frac{1}{2}gt^2$$

and
$$x_2 = u_2 \cos\theta_2 t$$
$$y_2 = u_2 \sin\theta_2 t - \frac{1}{2}gt^2$$
$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{u_2 \sin\theta_2 - u_1 \sin\theta_1}{u_2 \cos\theta_2 - u_1 \cos\theta_1} = m$$
$$Y = xX$$



which is a straight line.

Fig.

Q.10. A particle is moving in a circle of radius *r* with a uniform speed *v*. Find the rate at which the acceleration change.

Sol.
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} = -r\dot{\theta}^2\hat{r} \qquad \left[\because \dot{r} = 0, \frac{d\theta}{dt} = \dot{\theta} = \frac{v}{R}, \ddot{\theta} = 0\right]$$
$$= -R\frac{v^2}{R^2}\hat{r} = -\frac{v^2}{R}\hat{r}$$
$$\frac{d\vec{a}}{dt} = -\frac{v^2}{R}\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} = -\frac{v^2}{R}\frac{v}{R}\hat{\theta} = -\frac{v^3}{R}\hat{\theta}.$$

SECTION-C LONG ANSWER TYPE QUESTIONS

Q.1. Describe polar coordinates system in brief.

Ans.

Polar Coordinates System

It is an alternative coordinate system where very point on the plane is determined by a distance of the point from the reference origin (r) and an angle (θ) from a reference direction instead of x and y. Here r and θ are called (r, θ) polar coordinates and labeled as (r, θ). The distance of the point from the origin is called radial coordinate or radius r and the angle is called polar angle or azimuth or angular coordinate.

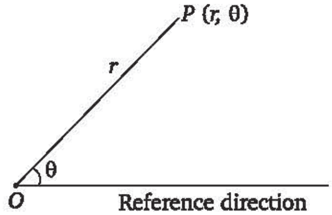


Fig. 1

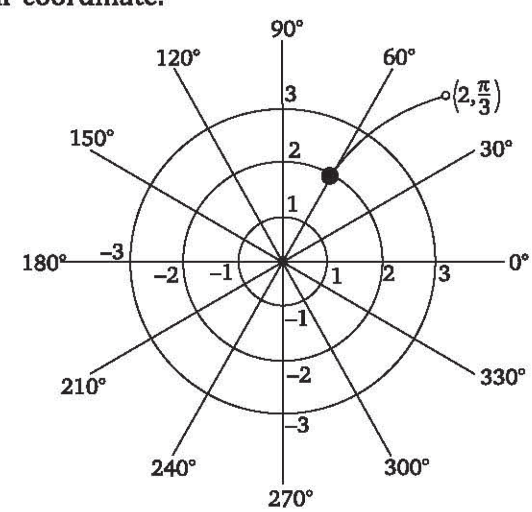


Fig. 2

A positive θ is measured counter clockwise from the axis. In order to plot the points on polar graph the r-coordinate is taken as the length of directed line segment from the pole (origin) and θ denotes the directions of r.

The range of r is from $0 \to \infty$, while θ ranges from 0 to π. (r, θ) Cartesian coordinates (x, y) and polar coordinates can be interconverted also. If we plot x and y axes associated with the origin of polar coordinate system, then

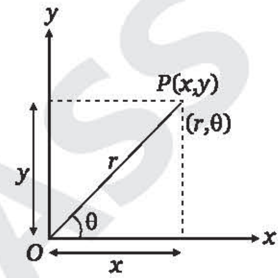


Fig. 3

$$\cos\theta = \frac{x}{r} \quad \Rightarrow \quad x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \quad \Rightarrow \quad y = r\sin\theta$$

Also, $$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$\Rightarrow$ $$r = \sqrt{x^2 + y^2}$$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

The above equations can be used to convert polar coordinates (r, θ) to Cartesian coordinates (x, y) and also Cartesian coordinate (x, y) to polar coordinates (r, θ).

Q.2. Define 3-D rectangular cartesian system. Distinguish between left and right handed cartesian system.

Ans. Three Dimensional (3-*D*) Rectangular Cartesian System

In three dimensional space, the rectangular Cartesian system is based on three mutually perpendicular coordinate axes, the x-axis, y-axis and z-axis. These three axes intersect at a common point, called the origin O of the system. In this system, a point in space is represented by three coordinates (x, y, z). These coordinates represent the distance of the point under reference from origin O along the x, y and z-axes respectively. To visualise, let us consider the origin O the point where the walls in the corner of a room meet the floor. The x-axis and y-axis are horizontal lines representing two sides of your room. The z-axis is the vertical line at the point of intersection of x and y-axis *i.e.*, the corner of the room.

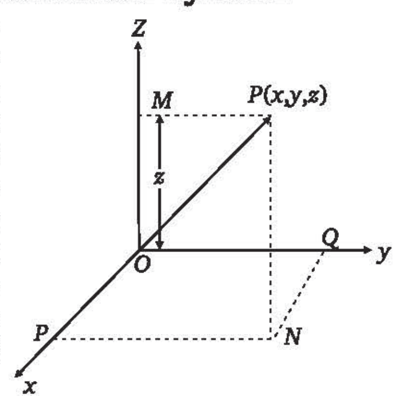


Fig. 1

Now, we consider a point $P(x, y, z)$ some where in room. Drop a perpendicular on z-axis. It meets on M. The distance of foot of perpendicular M from the origin is z-coordinate of P. Similarly drop a perpendicular from P on x-y plane. It will meet on the plane at N. From N again, drop perpendiculars on x and y axis and distances of foot of perpendicular from N on x and y axes *i.e.*, P and Q from the origin O represents x and y coordinates of the point P.

The rectangular Cartesian system may be of two types :

1. **Right Handed Coordinate System :** If you curl the fingers of your right hand from positive x-axis to positive y-axis and the thumb of your right hand points in the direction of positive z-axis, the resulting coordinate system is called right handed. A right handed system is shown in figure where x and y-axis are on the plane of the paper and z-axis is perpendicular to the plane of paper directed outwards.

2. **Left Handed Coordinate System :** If the three axes are chosen in such a way that curling the fingers clockwise from x-axis to y-axis results the direction of thumb along z-axis. A left handed system is shown in figure where z-axis is normal to the plane of paper directed inwards.

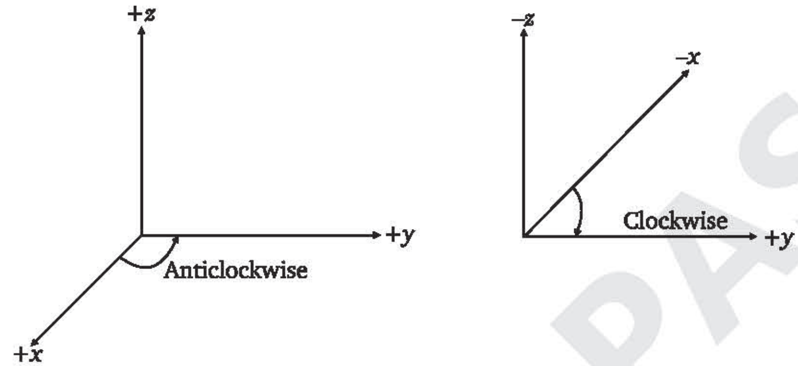


Fig. 2

The left and right handed coordinate systems are two equally valid mathematical universes. The left handed system can also be obtained by mirror reflection of the right handed system.

Let x, y and z-axes are represented by three unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$ pointing towards increasing x, y and z-coordinates from the origin. The position vector of point $P(x, y, z)$ can be written as

$$\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$$

Q.3. Show that the radical and tangential components of acceleration in two-dimensional space are given as :

$$a_r = \ddot{r} - r\dot{\theta}^2 = \frac{x\ddot{x} + y\ddot{y}}{\sqrt{x^2 + y^2}}$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{x\ddot{y} + \ddot{x}y}{\sqrt{x^2 + y^2}}$$

Sol.

$$x = r\cos\theta;\ y = r\sin\theta \qquad \text{...(1)}$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

$$\ddot{x} = \ddot{r}\cos\theta - \dot{r}\sin\theta\,\dot{\theta} - \dot{r}\sin\theta\,\dot{\theta} - r\cos\theta\,\dot{\theta}^2 - r\sin\theta\,\ddot{\theta}$$

$$= (\ddot{r} - r\dot{\theta}^2)\cos\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin\theta \qquad \text{...(2)}$$

Similarly

$$\ddot{y} = (\ddot{r} - r\dot{\theta}^2)\sin\theta + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\cos\theta \qquad \text{...(3)}$$

From equation (2) and (3)

$$\ddot{x}\cos\theta = (\ddot{r} - r\dot{\theta}^2)\cos^2\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin\theta\cos\theta \qquad \text{...(4)}$$

$$\ddot{y}\sin\theta = (\ddot{r} - r\dot{\theta}^2)\sin^2\theta + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin\theta\cos\theta \qquad \text{...(5)}$$

Adding equation (4) and (5)

$$\ddot{x}\cos\theta + \ddot{y}\sin\theta = \ddot{r} - r\dot{\theta}^2$$

$$a_r = \ddot{r} - r\dot{\theta}^2 = \ddot{x}\frac{x}{r} + \ddot{y}\frac{y}{r}$$

$$a_r = \frac{x\ddot{x} + y\ddot{y}}{\sqrt{x^2 + y^2}}$$

Similarly, $$\ddot{x}\sin\theta = (\ddot{r} - r\dot{\theta}^2)\sin\theta\cos\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin^2\theta$$

$$\ddot{y}\cos\theta = (\ddot{r} - r\dot{\theta}^2)\sin\theta\cos\theta + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\cos^2\theta$$

Now $$\ddot{y}\cos\theta - \ddot{x}\sin\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

$$|a_\theta| = \dot{r}\dot{\theta} + 2\dot{r}\dot{\theta} = \ddot{y}\frac{x}{r} - \ddot{x}\frac{y}{r} = \frac{x\ddot{y} - y\ddot{x}}{\sqrt{x^2 + y^2}}$$

Q.4. Describe orthogonal curvilinear coordinates in detail.

Ans.

Orthogonal Curvilinear Coordinates

For solving partial differential equations related to various physics problems, it is often necessary to rewrite them in suitable coordinates instead of Cartesian coordinates to find simpler solutions. Enormous simplifications can be achieved if all the boundaries of the problem correspond to coordinate surfaces which are generated by holding one coordinate constant and varying the other two. Accordingly many special coordinate systems have been developed to solve problems in particular geometries. The most useful of these systems are orthogonal i.e., at any point of space, the vectors aligned with the three coordinate directions are mutually perpendicular. In general, the variation of a single coordinate will generate a curve in space rather than a straight line, and hence it is called curvilinear.

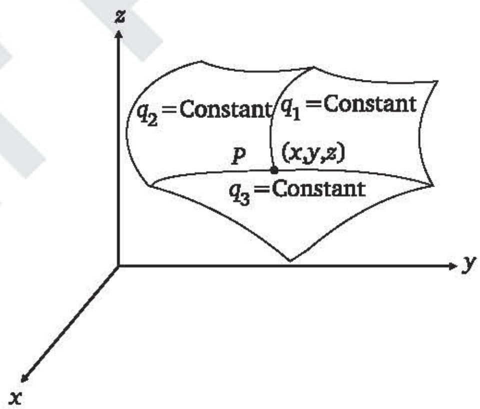


Fig.

In a cartesian system the position of point $P(x, y, z)$ is determined by three mutually perpendicular planes *i.e.,* x = constant, y = constant and z = constant.

Let us consider three other families of surfaces described as q_1 = constant, q_2 = constant and q_3 = constant. The surface of these families need not be parallel and also planes, but intersect at point *P*. The values of q_1, q_2, q_3 for these three surfaces intersecting at *P* are called curvilinear coordinates *e.g., P*. The three new surfaces are often called coordinate surface or curvilinear surfaces.

If the three coordinates surface q_1 = constant, q_2 = constant and q_3 = constant are mutually perpendicular at every point $P(x, y, z)$ then (q_1, q_2, q_3) are called orthogonal curvilinear coordinates. Also (x, y, z) can be expressed in terms of (q_1, q_2, q_3) and *vice-versa*.

$$\left.\begin{aligned} x &= x(q_1, q_2, q_3) \\ y &= y(q_1, q_2, q_3) \\ z &= z(q_1, q_2, q_3) \end{aligned}\right\}$$

or

$$\left.\begin{aligned} q_1 &= q_1(x, y, z) \\ q_2 &= q_2(x, y, z) \\ q_3 &= q_3(x, y, z) \end{aligned}\right\}$$

With each family of surface q_i = constant a unit vector $\hat{u}_i$ can be associated in the direction of increasing q_i and normal to surface q_i = constant

The partial differentiation of the equation above

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3$$

$$dy = \frac{\partial y}{\partial q_1} dq_1 + \frac{\partial y}{\partial q_2} dq_2 + \frac{\partial y}{\partial q_3} dq_3$$

$$dz = \frac{\partial z}{\partial q_1} dq_1 + \frac{\partial z}{\partial q_2} dq_2 + \frac{\partial z}{\partial q_3} dq_3$$

The square of the distance between two neighbouring points.

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= \sum_i \sum_j h_{ij}^2 dq_i dq_j$$

where coefficients h_{ij} are given as

$$h_{ij}^2 = \left(\frac{\partial x}{\partial q_i}\right)\left(\frac{\partial x}{\partial q_j}\right) + \left(\frac{\partial y}{\partial q_i}\right)\left(\frac{\partial y}{\partial q_j}\right) + \left(\frac{\partial z}{\partial q_i}\right)\left(\frac{\partial z}{\partial q_j}\right)$$

For a orthogonal coordinate system

$$h_{ij} = 0, i \neq j$$

If we take $h_{ij} = h_i$, then

$$ds^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$$

h_1, h_2, h_3 are called scale factors.

The distance between any two points on a coordinate line is called the line element. When the variation is limited to any given q_i with other $q's$ constant, then

$$ds_i = h_i dq_i$$

The three curvilinear coordinates q_1, q_2, q_3 need not be lengths. Also scale factors h_1, h_2, h_3 may depend on $q's$ and have dimensions. However, the product $h_i dq_i$ will have the dimensions of length.

The three possible surface element in an orthogonal system are

$$ds_{ij} = ds_i ds_j = h_i h_j dq_i dq_j \ (i, j = 1, 2, 3 \text{ and } i \neq j)$$

and volume element

$$dV = ds_1 ds_2 ds_3 = h_1 h_2 h_3 dq_1 dq_2 dq_3$$

Q.5. Derive the expressions for velocity and acceleration components in a polar coordinate system.

Ans.
Expression for Velocity and Acceleration Components

In polar coordinates, the position of a particle P is determined by value of the radial distance to the origin r and the angle that the radial line makes with an arbitrary fixed line, such as x-axis. The trajectory of the particle is determined by knowing both r and θ as a function of times t *i.e.*, $r(t)$ and $\theta(t)$. The directions of increasing r and θ are defined by unit vectors $\hat{r}$ and $\hat{\theta}$.

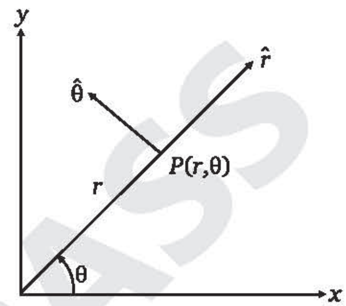


Fig. 1

The position vector of the particle can be written as

$$\vec{r} = r\hat{r} \quad \ldots(1)$$

Since the vector $\hat{r}$ and $\hat{\theta}$ are clearly different from point to point, their variation will also have to be considered while calculating velocity and acceleration over an infinitesimal interval of time, the coordinate of point P from (r, θ) to $(r + dr, \theta + d\theta)$ as shown in diagram below :

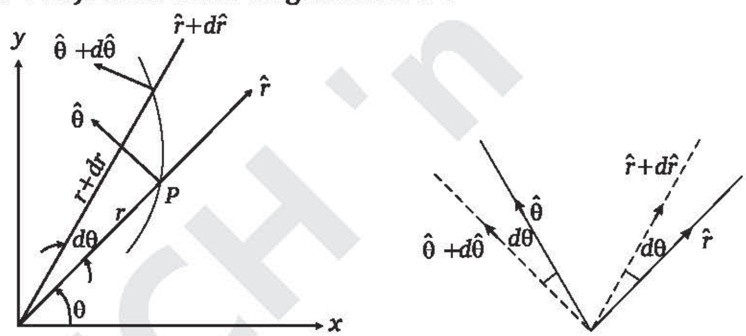


Fig. 2

One can see that $\hat{r}$ and $\hat{\theta}$ do not change when coordinate r charges.

i.e.,
$$\frac{d\hat{r}}{dr} = 0 \quad \text{and} \quad \frac{d\hat{\theta}}{d\theta} = 0 \quad \ldots(2)$$

On the other hand, when θ charges from θ to $\theta + d\theta$, the vector $\hat{r}$ and $\hat{\theta}$ are rotated by an angle $d\theta$. From the figure

$$\left.\begin{aligned} d\hat{r} &= d\theta\,\hat{\theta} \\ \text{and} \quad d\hat{\theta} &= -d\theta\,\hat{r} \end{aligned}\right\} \quad \ldots(3)$$

This is because their magnitudes in the limit are equal to the unit vector as radius time $d\theta$ in radians. Dividing by $d\theta$

$$\frac{d\hat{r}}{d\theta} = \hat{\theta} \quad \text{and} \quad \frac{d\hat{\theta}}{d\theta} = -\hat{r} \quad \ldots(4)$$

Multiplying both sides by $\dot{\theta}$

$$\left(\frac{d\hat{r}}{d\theta}\right)\cdot\left(\frac{d\theta}{dt}\right)=\dot{\theta}\,\hat{\theta} \quad \text{and} \quad \left(\frac{d\hat{\theta}}{dt}\right)\cdot\left(\frac{d\theta}{dt}\right)=-\hat{r}\,\dot{\theta}$$

$$\frac{d\hat{r}}{dt}=\dot{\theta}\,\hat{\theta} \quad \text{and} \quad \frac{d\hat{\theta}}{dt}=-\hat{r}\,\dot{\theta} \qquad \text{...(5)}$$

Mathematically it can be done as follows :

$$\left.\begin{aligned}\hat{r}&=\cos\theta\,\hat{i}=\sin\theta\,\hat{j}\\ \hat{\theta}&=\cos(90+\theta)\,\hat{i}+\sin(90+\theta)\,\hat{j}=-\sin\theta\,\hat{i}+\cos\theta\,\hat{j}\end{aligned}\right\} \qquad \text{...(6)}$$

Differentiating with respect to r

$$\frac{d\hat{r}}{dr}=0 \quad \text{and} \quad \frac{d\hat{\theta}}{dr}=0 \qquad \text{...(7)}$$

Now differentiating with respect to θ

$$\left.\begin{aligned}\frac{d\hat{r}}{d\theta}&=-\sin\theta\,\hat{i}+\cos\theta\,\hat{j}=\hat{\theta}\\ \frac{d\hat{\theta}}{d\theta}&=-\cos\theta\,\hat{i}-\sin\theta\,\hat{j}=-\hat{r}\end{aligned}\right\} \qquad \text{...(8)}$$

Now from equation (1), the velocity of the particle

$$\vec{v}=\dot{\vec{r}}=\dot{r}\,\hat{r}+r\,\dot{\hat{r}}$$

$$\vec{v}=\dot{r}\,\hat{r}+r\,\dot{\theta}\,\hat{\theta}=v_r\,\hat{i}+v_\theta\,\hat{\theta}$$

Here $v_r=\dot{r}$ is the radical velocity and $v_\theta=r\,\dot{\theta}$ is called tangential/circumferential velocity component.

Total velocity $\qquad v=\sqrt{v_r^2+v_\theta^2}$

The acceleration can be obtained by differentiating velocity with respect to time.

$$\vec{a}=\dot{\vec{v}}=\ddot{r}\,\hat{r}+\dot{r}\,\frac{d\hat{r}}{dt}+\dot{r}\,\dot{\theta}\,\hat{\theta}+r\,\ddot{\theta}\,\hat{\theta}+r\,\dot{\theta}\,\frac{d\hat{\theta}}{dt}$$

$$=\ddot{r}\,\hat{r}+\dot{r}\,\dot{\theta}\,\hat{\theta}+\dot{r}\,\dot{\theta}\,\hat{\theta}+r\,\ddot{\theta}\,\hat{\theta}+r\,\dot{\theta}\,(-\hat{r}\,\dot{\theta})$$

$$\vec{a}=(\ddot{r}-r\,\dot{\theta}^2)\,\hat{r}+(r\,\ddot{\theta}+2\dot{r}\,\dot{\theta})\,\hat{\theta}=a_r\,\hat{r}+a_\theta\,\hat{\theta}$$

$$a_r=\ddot{r}-r\,\dot{\theta}^2 \rightarrow \text{radical acceleration}$$

$$a_\theta=r\,\ddot{\theta}+2\,\dot{r}\,\dot{\theta} \rightarrow \text{tangential acceleration}$$

Total acceleration $a = \sqrt{a_r^2 + a_\theta^2}$

In two dimensional polar coordinate system, the equations of motions of the particle can be written as

$$F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)$$

$$F_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

Q.6. Explain the non-inertial coordinate system or frame of reference.

Ans.

Non-inertial Coordinate System

A coordinate system or frame of reference in which Newton's laws of motion are valid is called an inertial frame of reference. All frames of references moving uniformly relative to an inertial frame are also inertial. But when a frame of reference is accelerated with respect to an inertial frames, the forms of physical laws such as Newton second law becomes different. Such a coordinate system having an acceleration relative to an inertial frame is called non-inertial frames of reference. Some of the examples of non-inertial frames are :

(i) An accelerated train

(ii) A car tarting a round turn with constant speed.

Let a reference from S' moves with a constant acceleration a_0 with respect to an inertial frame S. The acceleration of a particle of mass m in S frame is a.

The acceleration of particle as observed by observer in frame of reference S' is

$$\vec{a}' = \vec{a} - \vec{a}_0$$

$$m\vec{a}' = m\vec{a} - m\vec{a}_0$$

$$a' = \frac{F - ma_0}{m}$$

Thus, Newtons second law of motion $\vec{a} = \frac{\vec{F}}{m}$ is not valid in a non-inertial frame as the acceleration of the frame enters into the equation of motion for a particle.

In such frame an extra term $-m\vec{a}_0$ has been added to the sum of all forces to write Newtons second law.

Pseudo Force

In order to retain the terminology of Newtons second law even in non-inertial frame of references, for computing the total force on the particle, an imaginary force $-ma_0$ is also added with all real forces acting on the body. Such correction terms $-ma_0$ are called pseudo forces. This pseudo force is taken into consideration for discussing the motion of a particle in a non-inertial frame and still retaining Newtons second law. Pseudo forces are also called fictitious forces.

$$m\vec{a}' = \vec{F} - m\vec{a}_0 = \vec{F} + \vec{F}_s$$

where $\vec{F}_s = -m\vec{a}_0$ is pseudo force.

Total force = True force + pseudo force

Total acceleration = True acceleration - acceleration of non inertial frame

Centrifugal force is an interesting example of pseudo force. A uniformly rotating frame has a centripetal acceleration, therefore it is also a non-inertial frame. Coriolis force is also a fictitious force which acts on the particle when it is in motion relative to a rotating frame.

Q.7. A particle moves with $\dot{\theta} = \omega$ and $r = r_0 e^{\omega t}$

(i) sketch the path

(ii) find speed of the particle as a function of θ

(iii) radial and tangential acceleration.

(iv) show that the tangential acceleration is $\frac{dv}{dt}$.

Sol. (i) $\theta = \omega t$

$\therefore \quad r = r_0 e^{\theta}$

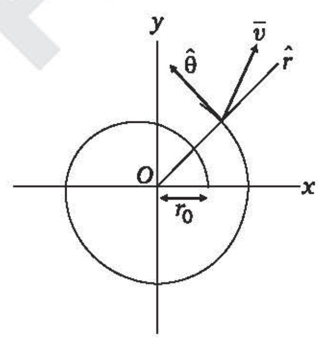


Fig.

(ii) $\bar{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{r}$

$r = r_0 e^{\theta}$

$\dot{r} = r_0 e^{\theta}\dot{\theta} = r_0 \omega e^{\theta}$

$\therefore \quad v_r = \dot{r} = r_0 \omega e^{\theta} v_{\theta} = r\dot{\theta} = r_0 e^{\theta}\omega$

$\therefore \quad \bar{v} = r_0 \dot{e}^{\theta}\,\omega(\hat{r} + \hat{\theta})$

$\therefore \quad |\bar{v}| = r_0 e^{\theta}\omega\sqrt{2}$

(iii) $a_r = \ddot{r} - r\dot{\theta}^2$

$\dot{r} = r_0 \omega e^{\theta}$

$\ddot{r} = r_0 \omega e^{\theta}\dot{\theta} = r_0 \omega^2 e^{\theta}$

$\therefore \quad a_r = r_0 \omega^2 e^{\theta} - r_0 e^{\theta}\omega^2 = 0$

$a_{\theta} = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

$\dot{\theta} = \omega \Rightarrow \ddot{\theta} = 0$

$\therefore \quad a_{\theta} = 2\dot{r}\dot{\theta} = 2r_0 \omega e^{\theta}\omega = 2r_0 \omega^2 e^{0}$

$\therefore \quad a_t = 2r_0 \omega^2 e^{\omega t}\cos 45° = \sqrt{2} r_0\,\omega^2\,e^{\omega t}$

(iv) $v = \sqrt{2}\,\omega r_0\,e^{\theta}$

$$\frac{dv}{dt} = \sqrt{2}\,\omega r_0 e^{\omega t}\omega = \sqrt{2}\omega^2 r_0 e^{\omega t} = a_t.$$

❑

UNIT-IV

Introduction to Tensors

SECTION-A VERY SHORT ANSWER TYPE QUESTIONS

Q.1. What are tensor quantities?

Ans. A physical quantity which is neither vector nor scalar is known as tensor quantity. A tensor field has a tensor corresponding to each point space. An example is the stress on a material such as a construction beam in a bridge. Stress is a tensor quantity.

Q.2. Define Einstein summation convention.

Ans. **Einstein Summation Convention :** If any index is repeated in a single term once as a subscript and once as a superscript, then the single term represents the summation over all admissible values of the repeated index. This convention is called Einstein summation convention.

$$\sum_j a_j x^j = a_j x^j$$

$$\sum_i \sum_j \alpha_{ij} x^i x^j = a_{ij} x^i x^j$$

Q.3. Write the law of transformation of tensor A^{ij}_k.

Ans. $A'^{ij}_{k} = \dfrac{\partial x'^i}{\partial x^a} \cdot \dfrac{\partial x'^j}{\partial x^b} \cdot \dfrac{\partial x^c}{\partial x'^k} A^{ab}_c.$

Q.4. What is meant by invariant tensor?

Ans. **Invariant Tensor :** When a physical quantity have a value A in coordinates x^i and A' under a transformation to a new set of coordinates x'^i and then if

$$A' = A$$

then the quantity A is called a scalar or an invariant or a tensor of rank zero.

Q.5. If A^{ij} are n^2 quantities, show that $\delta^i_j \cdot A^{jk} = A^{ik}$.

Ans. $\delta^i_j \cdot A^{jk} = \delta^i_1 A^{1k} + \delta^i_2 A^{2k} + \delta^i_3 A^{3k} + \ldots + \delta^i_n A^{nk}$

$= A^{ik}$ (The only term which will be non zero is the one where $i = n$).

Q.6. What is the fourier series and its applications?

Ans. A fourier series is a specific type of infinite mathematical series that involves trigonometric functions. Fourier series are the ones which are used in applied mathematics and especially in the field of physics and electronics to express periodic functions such as those that comprise communications signal wave forms.

Q.7. Show that if a tensor is antisymmetric with respect to a pair of indices in one coordinate system, it is same in every system.

Ans. $A_{ij} = -A_{ji}$ in coordinate system x^i

$$A'_{ij} = \frac{\partial x^{\alpha}}{\partial x'^{i}} \cdot \frac{\partial x^{\beta}}{\partial x'^{j}} A_{\alpha\beta}$$

$$= -A_{\beta\alpha} \frac{\partial x^{\beta}}{\partial x'^{j}} \cdot \frac{\partial x^{\alpha}}{\partial x'^{i}} = -A'_{ji}$$

Q.8. What is meant by four vector?

Ans. In special relativity, a four vector is an object with four components, which transform in a specific way under Lorentz transformation.

Q.9. Show that the velocity of a fluid at any point is a contravariant vector.

Sol. The velocity of fluid at any point in coordinate system x^i has component $\frac{dx^i}{dt}$ and x'^i has $\frac{dx'^i}{dt}$. Now

$$\frac{dx'^i}{dt} = \frac{\partial x'^i}{\partial x^1} \cdot \frac{dx^1}{dt} + \frac{\partial x'^i}{\partial x^2} \cdot \frac{dx^2}{dt} + \ldots + \frac{\partial x'^i}{\partial x^n} \cdot \frac{dx^n}{dt}$$

$$= \frac{\partial x'^i}{\partial x^j} \cdot \frac{dx^j}{dt}$$

The equation satisfies contravariant law of transformation and hence velocity at any point is a contravariant vector.

Q.10. Define mixed tensor of rank 2.

Ans. Mixed Tensor or Rank 2 : If $(n)^2$ quantities A^i_j in a system of variable x^i are related to other $(n)^2$ quantities A'^i_j in another system of variables x'^i by transformation equation.

$$A'^i_j = \frac{\partial x'^i}{\partial x^{\alpha}} \cdot \frac{\partial x^{\beta}}{\partial x'^j} \cdot A^{\alpha}_{\beta}$$

then the quantities A^i_j are said to be the component of a mixed tensor of second rank.

Kronecker delta δ^{μ}_{υ} is a mixed tensor of second rank.

Q.11. Write the following in brief form using summation convention :

$$a_1 x^1 y^3 + a_2 x^2 y^3 + \ldots + a_n x^n y^3$$

Ans. In the given series superscript of $y = 3$ is same in all terms while the subscript of a and superscript of x vary from 1 to n. Therefore the above series can be written as $a_i x^i y^3$.

SECTION-B SHORT ANSWER TYPE QUESTIONS

Q.1. What is Kronecker Delta?

Ans. **Kroncker Delta**

Kronecker delta symbol $\delta^{\mu}_{\nu} = \begin{cases} 1 \text{ if } \mu = \nu \\ 0 \text{ if } \mu \neq \nu \end{cases}$

It is very important in tensor analysis. The properties of Kronecker delta are as follows :

1. $\delta^1_1 = \delta^2_2 = \ldots = \delta^{\mu}_{\mu} = 1$
2. $\delta^1_2 = \delta^3_2 = \ldots = \delta^{\nu}_{\mu} = 0 \, (\mu \neq \nu)$
3. $\delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 + \ldots + \delta^n_n = n$
4. $A^j \delta^i_j = A^i$
5. $\dfrac{\partial x^i}{\partial x^j} = \delta^i_j$ $\quad \left(\because \dfrac{\partial x^i}{\partial x^j} = 1 \text{ if } i = j \text{ and } \dfrac{\partial x^i}{\partial x^j} = 0 \text{ if } i \neq j \right)$
6. $\dfrac{\partial x^i}{\partial x'^j} \cdot \dfrac{\partial x'^j}{\partial x^k} = \dfrac{\partial x^i}{\partial x^k} = \delta^i_k$

   $\dfrac{\partial x'^i}{\partial x^j} \cdot \dfrac{\partial x^j}{\partial x'^k} = \dfrac{\partial x'^i}{\partial x'^k} = \delta^i_k$
7. $\delta^i_j \, \delta^j_k = \delta^i_k$.

Q.2. Distinguish between symmetric and antisymmetric tensors.

Sol. **Symmetric and Antisymmetric Tensors**

A tensor A^{ij} is said to be symmetric tensor is

$$A^{ij} = A^{ji}$$

i.e., It remains unaltered with interchange of indices.

A symmetric tensor remains unaltered by coordinates transformation

$$A'^{ij} = A'^{ji}$$

$$A'^{ij} = \frac{\partial x'^i}{\partial x^{\alpha}} \cdot \frac{\partial x'^j}{\partial x^{\beta}} A^{\alpha\beta} = \frac{\partial x'^i}{\partial x^{\beta}} \cdot \frac{\partial x'^j}{\partial x^{\alpha}} A^{\beta\alpha}$$

$$= \frac{\partial x'^j}{\partial x^{\beta}} \cdot \frac{\partial x'^i}{\partial x^{\alpha}} A^{\beta\alpha} = A'^{ji}$$

Similarly a tensor A^{ij} is said to be antisymmetric or skew symmetric if

$$A_{ij} = -A_{ji}$$

The anti-symmetric property also remain unaltered by coordinate transformation *i.e.*,

$$A'^{ij} = -A'^{ji}$$

$$A'^{ij} = -\frac{\partial x'^i}{\partial x^\alpha} \cdot \frac{\partial x'^j}{\partial x^\beta} A^{\alpha\beta} = -\frac{\partial x'^i}{\partial x^\beta} \cdot \frac{\partial x'^j}{\partial x^\alpha} A^{\beta\alpha}$$

$$= -\frac{\partial x'^j}{\partial x^\beta} \cdot \frac{\partial x'^i}{\partial x^\alpha} A^{\beta\alpha} = A'^{ji}$$

Q.3. What are Levi-Civita Symbols? Explain briefly.

Ans.

Levi-Civita Symbol

Levi-Civita symbol in three dimensional space is a tensor of rank-3 and is denoted by $\varepsilon_{\mu\nu\sigma}$ while in a four dimensional space, it is a tensor of rank 4 denoted by $\varepsilon_{\mu\nu\sigma\delta}$.

It is also called Epsilon tensor or alternate tensor or permutation tensor.

In three dimensional space, Levi-Civita symbol $\varepsilon_{\mu\nu\sigma}$ is a quantity which is anti-symmetric in all its indices.

$$\varepsilon_{\mu\nu\sigma} = \begin{cases} +1 \text{ if } \mu, \nu, \sigma \text{ is an even permutation of } (1,2,3) \\ -1 \text{ if } \mu, \nu, \sigma \text{ is an odd permutation of } (1,2,3) \\ -0 \text{ otherwise (contain two or more repeated indices} \end{cases}$$

Levi-Civita tensor and anti-symmetric in every pair of indices

$$\varepsilon_{\mu\nu\sigma} = -\varepsilon_{\nu\mu\sigma} = +\varepsilon_{\nu\sigma\mu} = -\varepsilon_{\sigma\nu\mu} = +\varepsilon_{\sigma\mu\nu}$$

Four Vectors : In relativity, an event is defined by space time coordinates (x, y, z, t). When we look at the event from another moving coordinate system, both the space and time coordinate are no more independent, but they are dependent on each other. Four vector is a representation of an event with four components which transfrom from one coordinate system to another. A four vector A is a vector with notations

$$A = (A^0, A^1, A^2, A^3) \text{ or } (A^1, A^2, A^3, A^4)$$

$$= A^\mu \quad \text{where } \mu = 0,1,2,3 \quad \text{or} \quad 1,2,3,4$$

Four vector transform like tensors.

Q.4. Write all the terms of $a_{ij}x^i x^j$ where $i_{ij} = 1, 2, 3$.

Sol.

$$a_{ij}x^i x^j = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij}x^i x^j$$

$$= \sum_{i=1}^{3} (a_{i1}x^i x^1 + a_{i2}x^i x^2 + a_{i3}x^i x^3)$$

$$= a_{11}x^1 x^1 + a_{21}x^2 x^1 + a_{31}x^3 x^1 + a_{12}x^1 x^2$$

$$+ a_{22}x^2 x^2 + a_{32}x^3 x^2 + a_{13}x^1 x^3 + a_{23}x^2 x^3 + a_{33}x^3 x^3$$

$$= a_{11}(x^1)^2 + a_{22}(x^2)^2 + (a_{33}x^3)^2 + (a_{12} + a_{21})x^1 x^2$$

$$+ (a_{23} + a_{32})x^2 x^3 + (a_{31} + a_{13})x^3 x^1$$

Q.5. What is a Fourier series? Also write about periodic, even and odd functions.

Ans.

Introduction to Fourier Series

Fourier series are used in the analysis of periodic functions. You must have come across many of the phenomena studied in science which are periodic in nature as the current and voltage in an alternating current circuit. Various waveforms such as square, saw tooth, triangle etc. occurring in electronics can be analysed by Fourier series. These periodic functions can be analysed into their constituent components (Fundamentals and harmonics) by a process called Fourier analysis, Essentially Fourier series arise from the practical task of representing a given periodic function $f(x)$ in terms of cosine and sine functions. This is done by adding more and more trigonometrical functions together. The sum of these special trigonometric functions is called the Fourier series.

Periodic Functions : A function $f(x)$ is called periodic if there is some positive number T such that

$$f(x+T)=f(x) \text{ for all } x$$

The number T is called period of the function $f(x)$. The graph of such functions can be obtained by periodic repetition of its graph in any interval of length T. The most familiar periodic functions are sine and cosine functions. Sin x and cos x has a period of 2π.

If two functions $f(x)$ and $g(x)$ have period T, then the function $h(x)=af(x)+bg(x)$ has also period T with a and b constants.

Some of the examples of functions that are not periodic are

$$\tan x, x, x^2, x^3, e^x, \cos x, \ln x \text{ etc.}$$

Even and Odd Functions : A function $f(x)$ is known to be an even function if

$$f(-x)=f(x)$$

The function $f(x)$ is known to be an odd function if

$$f(-x)=-f(x).$$

Graphically, even functions have symmetry about the y-axis, whereas odd functions have symmetry around the origin.

Q.6. Express the relationship between cartesian and spherical polar coordinates of a function.

Sol. The relation between cartesian coordinates (x, y, z) and spherical polar coordinates (r, θ, ϕ) are

$$x=r\sin\theta\cos\phi$$

$$y=r\sin\theta\sin\phi$$

$$z=r\cos\theta$$

The inverse transformation is

$$r=\sqrt{x^2+y^2+z^2}$$

$$\theta=\tan^{-1}\left[\frac{(x^2+y^2)^{1/2}}{2}\right]$$

$$\phi=\tan^{-1}\frac{y}{x}$$

Q.7. Show that the Kronecker delta δ^i_j is a mixed tensor of rank two. Also show that it is an invariant.

Sol. Let δ^i_j be Kronecker delta in coordinate system x^i and δ'^i_j be Kronecker delta in coordinate system x'^i

Now $$\delta^i_j = \frac{\partial x^i}{\partial x^j}$$

Then $$\delta'^i_j = \frac{\partial x'^i}{\partial x'^j} = \frac{\partial x'^i}{\partial x^k} \cdot \frac{\partial x^k}{\partial x'^j}$$

$$= \frac{\partial x'^i}{\delta x^k} \cdot \frac{\partial x^k}{\partial x^l} \cdot \frac{\partial x^l}{\partial x'^j} = \frac{\partial x'^i}{\partial x^k} \cdot \frac{\partial x^l}{\partial x'^j} \cdot \delta^k_l$$

It show that δ^i_j is a mixed tensor of rank 2.

Also $$\delta'^i_j = \frac{\partial x'^i}{\partial x'^j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$\therefore$ $$\delta'^i_j = \delta^i_j$$

Hence, it is an invariant.

Q.8. If $a_{ij} x^i x^j = 0$ for all values of variables $x^1, x^2 \ldots . x^n$ then prove that $a_{ij} + a_{ji} = 0$.

Sol. $a_{ij} x^i x^j = 0$

Differentiating partially with respect to x^k

$$a_{ij} x^i \frac{\partial x^j}{\partial x^k} + a_{ij} \cdot \frac{\partial x^i}{\partial x^k} x^j = 0$$

$$a_{ij} \, x^i \delta^j_k + a_{ij} \delta^i_k x^j = 0$$

$$a_{ik} x^i + a_{kj} x^j = 0$$

Differentiating again partially with respect to x^l

$$a_{ik} \frac{\partial x^i}{\partial x^l} + a_{kj} \frac{\partial x^j}{\partial x^l} = 0$$

$$a_{ik} \delta^i_l + a_{kj} \delta^j_l = 0$$

$$a_{lk} + a_{kl} = 0$$

Replacing free indices l and k by i and j respectively

$$a_{ij} + a_{ji} = 0$$

Q.9. Show that the transformations of a contravariant vector posses the group property.

Ans. The transformation of a contravariant tensor possess the group or transitive property if the resultant transformation of a type is again a transformation of the same type.

Let A^i be the components of a contravariant tensor in coordinate system x^i. If B^i be the components of the same contravariant tensor when coordinates x^i are transformed to y^i.

Then $$B^i = \frac{\partial y^i}{\partial x^j} A^j \quad ...(1)$$

Now, if C' be the components of same contravariant tensor when the coordinates y^i are transformed to z^i. Then

$$C^i = \frac{\partial z^i}{\partial y^k} B^k \quad ...(2)$$

From equation (1) and (2)

$$C^i = \frac{\partial z^i}{\partial y^k} \cdot \frac{\partial y^k}{\partial x^j} A_j$$

$$= \frac{\partial z^i}{\partial x^j} \cdot A_j \quad ...(3)$$

Equation (3) is law transformation of contravariant tensor when coordinates x^i are transformed to z^i.

Hence transformation of contravariant tensor possess group property.

SECTION-C LONG ANSWER TYPE QUESTIONS

Q.1. Derive the expression for coordinate transformation for *n*-dimensional space.

Ans. **Coordinate Transformation for *n*-Dimensional Space**

In a three dimensional space, a point is represented by a set of three numbers (x, y, z), called the coordinate of that point. Similarly in *n*-dimensional space a point is represented by a set of *n*-variables as

$$(x_1, x_2, x_3, ..., x_n) \text{ or } (x^1, x^2, x^3, ..., x^n)$$

Here 1, 2, 3, ..., *n* denote variables and not the power of variables. This *n*-dimensional space is denoted by V_n and the coordinates of the point in *n*-dimensional space V_n can be described as above.

Let $(x^1, x^2, x^3, ..., x^n)$ and $(x'^1, x'^2, x'^3, ..., x'^n)$ be the coordinates of a point in two different frames of reference. The set of *n*-equations define a transformation of coordinates from one frame of reference to another.

$$x'^i = x'^i (x^1, x^2, x^3, ..., x^n) \qquad i = 1, 2, 3, ..., n \quad ...(1)$$

Differentiating above equation with respect to x^j.

$$dx'^i = \sum_j \frac{\partial x'^i}{\partial x^j} dx^j = \frac{\partial x'^i}{\partial x^j} dx^j \qquad ...(2)$$

It is assumed that each x^i is single valued, continuous and have continuous derivatives.

Equation (1) defines the transformation of coordinates from one frame of reference to another.

If the Jacobian of the transformation

$$\det\left|\frac{\partial x'^i}{\partial x^j}\right| = \begin{vmatrix} \frac{\partial x'^1}{\partial x'} & \frac{\partial x'^1}{\partial x^2} \cdots\cdots & \frac{\partial x'^1}{\partial x^n} \\ \vdots & & \\ \vdots & & \\ \frac{\partial x'^n}{\partial x^1} & \frac{\partial x'^n}{\partial x^2} \cdots\cdots & \frac{\partial x'^n}{\partial x^n} \end{vmatrix} \neq 0 \qquad ...(3)$$

Then transformation equations (1) can be solved for x^i in terms of x'^i

$$x^i = x^i\,(x'^1, x'^2 \ldots x'^n) \qquad ...(4)$$

Equations (1) and (4) enable us to pass from one coordinate system to another and *vice-versa*.

Q.2. Differentiate between contravariant and covariant tensors on the basis of transformation laws obeyed by them and hence show that the velocity of the fluid at any point is contravariant tensor of rank-1 and also explain rank-2.

Ans.

Contravariant Vector

Let a physical entity is characterised by *n*-functions A^i when expressed in the x^i coordinate system with $x^1, x^2, x^3, \ldots x^n$. If the quantity A^i are transformed to any other coordinate system $x'^1, x'^2, x'^3 \ldots x'^n$ according to rule

$$A'^i = \frac{\partial x'^i}{\partial x^\alpha} A^\alpha \qquad ...(1)$$

then the functions A^i are called components of a contravariant vector.

The above relation can be easily inverted to expression A^i in terms of A'^α. Multiplying both side of equation by $\left(\frac{\partial x^k}{\partial x'^i}\right)$

$$\frac{\partial x^k}{\partial x'^i} A'^i = \frac{\partial x^k}{\partial x'^i} \cdot \frac{\partial x'^i}{\partial x^\alpha} A^\alpha$$

$$\frac{\partial x^k}{\partial x'^i} A'^i = \delta^k_\alpha\, A^\alpha = A^k \qquad ...(2)$$

Replacing k by i and i by α

$$A^i = \frac{\partial x^i}{\partial x'^\alpha} A'^\alpha \qquad ...(3)$$

The contravariant vector is also called **contravariant tensor of rank 1**.

Covariant Vector

A set of n-quantities A_i which are the functions of n coordinates x^i $(x^1, x^2 \ldots x^n)$ are said to be the component of a covariant vector if they transform according to the rule under a charge of coordinates from x^i to x'^i.

$$A'_i = \frac{\partial x^\alpha}{\partial x'^i} A_\alpha \quad \ldots(4)$$

The inverse transformation can be written as :

$$A_i = \frac{\partial x'^\alpha}{\partial x^i} A'_\alpha \quad \ldots(5)$$

The contravariant vector is called a **covariant tensor of rank-1.**

Tensor of Rank-2

Let A^{ij} $(i, j = 1, 2, 3, \ldots n)$ be n^2 functions of coordinates $x^1, x^2, \ldots, x^n$ and if they transform to A'^{ij} in another coordinate system $x'^1, x'^2 \ldots x'^n$ according to transformation

$$A'^{ij} = \frac{\partial x'^i}{\partial x^\alpha} \cdot \frac{\partial x'^j}{\partial x^\beta} A^{\alpha\beta}$$

then A^{ij} are called component of a contravariant tensor of rank two.

Similarly if the transformation of A_{ij} $(i, j = 1, 2 \ldots n)$ takes place from set of coordinates $(x^1, x^2 \ldots x^n)$ to another coordinate system $x'^1, x'^2, \ldots x'^n$ by rule

$$A'_{ij} = \frac{\partial x^\alpha}{\partial x'^i} \cdot \frac{\partial x^\beta}{\partial x'^j} A_{\alpha\beta}$$

Then A_{ij} are said to be covariant tensor of rank two.

Q.3. State and explain Fourier Theorem.

Ans.

Fourier Theorem

This theorem was developed by the French mathematician J.B. Fourier around 1800. According to this theorem. "Any finite single valued periodic function, which is either continuous or possess finite number of discontinuities, can be expressed as a sum of a number of sinusoidal or simple harmonic vibrations, whose frequencies are integral multiples of that of given functions".

A finite single valued periodic function can be expanded mathematically as

$$f(t) = a_0 + a_1 \cos\omega t + a_2 \cos 2\omega t + \ldots + a_n \cos n\omega t$$
$$+ b_1 \sin\omega t + b_2 \sin 2\omega t + \ldots + b_n \sin n\omega t$$
$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t \quad \ldots(1)$$

The right hand side of the expression represents a series, called Fourier series of $f(t)$. The coefficients a_0, a_n, b_n are called Fourier coefficient. The processes determining these coefficients is called Fourier analysis.

Evaluation of Fourier Coefficients

Integrating both side of equation (1) over time period T

$$\int_0^T f(t)\,dt = a_0\int_0^T dt + a_1\int_0^T \cos\omega t\,dt + \ldots + a_n\int_0^T \cos n\,\omega t\,dt$$

$$+ b_1\int_0^T \sin\omega t\,dt + b_2\int_0^T \sin 2\omega t\,dt + \ldots + b_n\int_0^T \sin n\omega t\,dt$$

$$= a_0 T$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt \qquad \ldots(2)$$

To evaluate a_n, multiply both sides of equation (1) by $\cos n\omega t$ and integrate both sides between the limit $t = 0$ to $t = T$

$$\int_0^T f(t)\cos n\omega t\,dt = a_0\int_0^T \cos n\omega t\,dt + a_1\int_0^T \cos\omega t\cos n\omega t\,dt + \ldots$$

$$+ a_n\int_0^T \cos^2 n\omega t\,dt + b_1\int_0^T \sin\omega t\cos n\omega t\,dt + \ldots$$

$$+ b_n\int_0^T \sin n\omega t\cos n\omega t\,dt$$

$$= 0 + 0 + \ldots + a_n\left(\frac{T}{2}\right) + 0 + \ldots 0 = a_n\frac{T}{2}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt \qquad \ldots(3)$$

Thus by putting $n = 1, 2, 3, \ldots$, the values of coefficient $a_1, a_2 \ldots a_n$ may be obtained.

Similarly, to evaluate b_n, multiply both sides of equation (1) by $\sin\omega t$ and integrate both sides between the limits $t = 0$ to $t = T$.

$$\int_0^T f(t)\sin n\omega t\,dt = a_0\int_0^T \sin n\omega t\,dt + a_1\int_0^T \cos\omega t\sin n\omega t\,dt + \ldots$$

$$+ a_n\int_0^T \cos n\omega t\sin n\omega t\,dt + b_1\int_0^T \sin\omega t\sin n\omega t\,dt + \ldots$$

$$+ b_n\int_0^T \sin^2 n\omega t\,dt$$

$$= 0 + 0 + \ldots + 0 + 0 + \ldots + b_n\left(\frac{T}{2}\right)$$

$$b_n = \frac{2}{T}\int f(t)\sin n\omega t\,dt \qquad \ldots(4)$$

Thus by putting $n = 1, 2, 3, \ldots$, the values of coefficient $b_1, b_2, \ldots b_n$ may be obtained.

Conditions for Applicability of Fourier Theorem : A given function is capable of Fourier analysis if it satisfies the following restrictions :

(i) The function should be periodic.

(ii) The function should be single valued.

(iii) The function should always be finite.

(iv) The function should not have an infinite number of discontinuities.

Q.4. A covariant tensor has components $xy, 2y - z^2, xz$ in rectangular coordinates. Determine its components in spherical coordinates.

Sol.

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

Let $x = x^1, y = x^2, z = x^3$

$$r = x'^1, \theta = x'^2, \phi = x'^3$$

Also $A_1 = xy, A_2 = 2y - z^2, A_3 = xz$

$$A'_1, A'_2, A'_3 = ??$$

$$A_i = \frac{\partial x^j}{\partial x'^i} A_j \text{ for } i = 1, 2, 3$$

$$A'_1 = \frac{\partial x^1}{\partial x'^1} A_1 + \frac{\partial x^2}{\partial x'^1} A_2 + \frac{\partial x^3}{\partial x'^1} A_3$$

$$= \frac{\partial x}{\partial r} xy + \frac{\partial y}{\partial x}(2y - z^2) + \frac{\partial z}{\partial r} xz$$

$$= (\sin\theta \cos\phi)(r^2 \sin^2\theta \sin\phi \cos\phi) + (\sin\theta \sin\phi)$$

$$= (2r \sin\theta \sin\phi - r^2 \cos^2\theta) + \cos\theta\,(r^2 \sin\theta \cos\theta \cos\phi)$$

$$A'_2 = \frac{\partial x^1}{\partial x'^2} A_1 + \frac{\partial x^2}{\partial x'^2} A_2 + \frac{\partial x^3}{\partial x'^2} A_3$$

$$= \frac{\partial x}{\partial \theta} xy + \frac{\partial y}{\partial \theta}(2y - z^2) + \frac{\partial z}{\partial \theta} xz$$

$$= (r \cos\theta \cos\phi)(r^2 \sin^2\theta \sin\phi \cos\phi) + (r \cos\theta \sin\phi)$$

$$(2r \sin\theta \sin\phi - r^2 \cos^2\theta) + (-r \sin\theta)(r^2 \sin\theta \cos\theta \cos\phi)$$

$$A'_3 = \frac{\partial x^1}{\partial x'^3} A_1 + \frac{\partial x^2}{\partial x'^3} A_2 + \frac{\partial x^3}{\partial x'^3} A_3$$

$$= \frac{\partial x}{\partial \phi} xy + \frac{\partial y}{\partial \phi}(2y - z^2) + \frac{\partial z}{\partial \phi} xz$$

$$= (-r \sin\theta \sin\phi)(r^2 \sin^2\theta \sin\phi \cos\phi) + (r \sin\theta \cos\phi)$$

$$(2r \sin\theta \sin\phi - r^2 \cos^2\theta)$$

Q.5. Discuss the applications of Fourier theorem in detail.

Ans.

Applications of Fourier Theorem

The applications of Fourier Theorem can be defined as follows :

1. Square Wave

The figure 1 shows a square waveform representation curve. The displacement of the function is constant and positive from $t=0$ to $\frac{T}{2}$ and also constant and negative from $t=\frac{T}{2}$ to $t=T$.

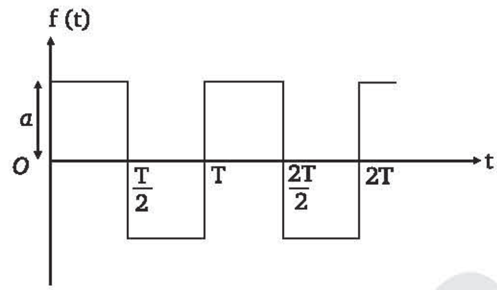


Fig. 1

One period of the square wave can be expressed as

$$\left.\begin{aligned} y=f(t)&=a \text{ from } t=0 \text{ to } t=\frac{T}{2}\left(0\le t\le\frac{T}{2}\right)\\ &=-a \text{ from } t=\frac{T}{2} \text{ to } t=T\left(\frac{T}{2}\le t\le T\right)\end{aligned}\right\} \quad ...(1)$$

From Fourier theorem, a complex periodic function may be expressed as a Fourier series.

$$f(t)=a_0+\sum_{n=1}^{\infty} a_n \cos n\omega t+\sum_{n=1}^{\infty} b_n \sin n\omega t$$

with

$$a_0=\frac{1}{T}\int_0^T f(t)\,dt$$

$$=\frac{1}{T}\int_0^{T/2} f(t)\,dt+\int_{T/2}^{T} f(t)\,dt$$

$$=\frac{1}{T}\int_0^{T/2} a\,dt+\int_{T/2}^{T} f(-a)\,dt$$

$$=\frac{a}{T}\left[\frac{T}{2}+\left(\frac{T}{2}-T\right)\right]=0 \quad ...(2)$$

$$a_n=\frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt$$

$$=\frac{2}{T}\int_0^{T/2} a\cos n\omega t\,dt+\int_{T/2}^{T} -a\cos n\omega t\,dt$$

$$=\frac{2a}{T}\left\{\left[\frac{\sin n\omega t}{n\omega}\right]_0^{T/2}-\left[\frac{\sin n\omega t}{n\omega}\right]_{T/2}^{T}\right\}$$

$$=\frac{2a}{n\omega T}\left\{\sin n\omega\frac{T}{2}-0-\sin n\omega T+\sin n\omega\frac{T}{2}\right\}$$

$$= \frac{2a}{2\pi n}\{\sin \pi n - \sin 2\pi n + \sin \pi n\} = 0 \qquad \text{...(3)}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\, dt$$

$$= \frac{2}{T}\int_0^{T/2} a \sin n\omega t\, dt + \int_{T/2}^{T} -a \sin n\omega t\, dt$$

$$= \frac{2a}{T}\left\{-\left[\frac{\cos n\omega t}{n\omega}\right]_0^{T/2} + \left[\frac{\cos n\omega t}{n\omega}\right]_{T/2}^{T}\right\}$$

$$= \frac{2a}{n\omega T}\left\{-\cos n\omega\frac{T}{2} + \cos 0 + \cos n\omega T - \cos n\omega\frac{T}{2}\right\}$$

$$= \frac{2a}{2\pi n}\{-\cos \pi n + 1 + \cos 2\pi n - \cos \pi n\}$$

$$= \frac{2a}{\pi n}\{1 - \cos \pi n\}$$

$$= \frac{2a}{\pi n}\{1 - (-1)^n\} \qquad \text{...(4)}$$

$$= 0 \text{ for even values of } n\ (n = 2, 4, 6, \ldots$$

$$= \frac{4a}{\pi n} \text{ for odd values of } n\ (n = 1, 3, 5, \ldots)$$

i.e.,

$$b_1 = \frac{4a}{\pi}, b_3 = \frac{4a}{3\pi} = \frac{4a}{3\pi}\ldots$$

$\therefore$

$$f(t) = \frac{4a}{\pi}\left[\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \ldots\right] \qquad \text{...(5)}$$

Thus a square wave is formed by the superposition of large number of simple harmonic vibrations, whose frequencies are odd integral multiple of the fundamental frequency. The amplitude of these harmonics are inversely proportional to their order.

2. Saw Tooth Waves

It is a triangular wave and because of its shape called saw-tooth wave. Its representation is shown in the figure 2.

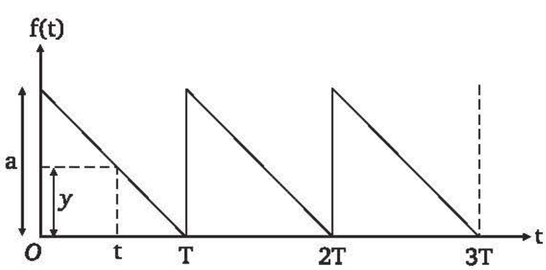


Fig. 2

It can be expressed as :

$$y = a \text{ at } t = 0$$

$$y = 0 \text{ at } t = T$$

From the property of similar triangle

$$\frac{y}{a} = \frac{T-t}{T} = 1 - \frac{t}{T}$$

$$y = a\left(1 - \frac{t}{T}\right) \qquad ...(6)$$

From Fourier series $\quad y = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$

Now, $\quad a_0 = \frac{1}{T}\int_0^T y\,dt = \frac{1}{T}\int_0^T a\left(1-\frac{t}{T}\right)dt$

$$= \frac{1}{T}\left[a\left(tr - \frac{t^2}{2T}\right)\right]_0^T = \frac{1}{T}\cdot\frac{aT}{2} = \frac{a}{2}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt$$

$$= \frac{2}{T}\int_0^T a\left(1-\frac{t}{T}\right)\cos n\omega t\,dt$$

$$= \frac{2a}{T}\int_0^T \cos n\omega t\,dt - \frac{2a}{T^2}\int_0^T t\cos n\omega t\,dt$$

$$= \frac{2a}{T}\left\{\left[\frac{\sin n\,\omega t}{n\omega}\right]_0^T - \frac{2a}{T^2}\left[t\,\frac{\sin n\omega t}{n\omega}\right]_0^T - \int_0^T \frac{\sin n\omega t}{n\omega}\,dt\right\}$$

$$= \frac{2a}{T}\left\{\frac{\sin n\,\omega t}{n\omega} - \frac{2aT}{T^2}\cdot\frac{\sin n\omega T}{n\omega} - \frac{2a}{T^2}\left[\frac{\cos n\omega T}{n^2\omega^2} - \frac{\cos 0}{n^2\omega^2}\right]\right\}$$

$$= 0 - 0 - 1 + 1 = 0$$

$$b_n = \frac{1}{T}\int_0^T y\sin n\omega t\,dt$$

$$= \frac{2}{T}\int_0^T a\left(1-\frac{t}{T}\right)\sin n\omega t\,dt$$

$$= \frac{2\pi}{T}\int_0^T \sin n\omega t\,dt - \frac{2a}{T^2}\int_0^T t\sin\omega t\,dt$$

$$= \frac{2a}{n\omega T}[\cos n\omega t]_0^T - \frac{2a}{T^2}\left\{\left[-t\,\frac{\cos n\omega t}{n\omega}\right]_0^T - \int_0^T -\frac{\cos n\omega t}{n\omega}\,dt\right\}$$

$$= \frac{a}{n\pi}[-\cos 2\pi n + \cot 0] - \frac{2a}{n\omega T^2}[-T\cos n\omega T + 0]$$

$$- \frac{2a}{n^2\omega^2 T^2}[\sin n\omega T - \sin 0]$$

$$= \frac{a}{n\pi}[-1+1] - \frac{2a}{2\pi n}[1-0] + \frac{2a}{4\pi^2 n^2}[0-0]$$

$$= \frac{a}{\pi n}$$

Substituting the values of Fourier coefficients

$$y = \frac{a}{2} + \frac{a}{\pi}\left[\sin \omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \ldots\right]$$

Thus, it be seen that all the even and odd harmonics of sine series are present with their amplitudes decreasing with the order of the harmonics.

Q.6. Prove that :

(i) $\delta_{il}\,\varepsilon_{ilm} = 0$

(ii) $\varepsilon_{iks}\,\varepsilon_{mps} = \delta_{im}\,\delta_{kp} - \delta_{kp} - \delta_{ip}\,\delta_{km} = 0$

Sol. **(i)** $\delta_{il}\varepsilon_{ilm} = \delta^i_l\varepsilon_{ilm}$

If $i = l$ $\quad = \delta^i_i\,\varepsilon_{iim} = 1 \times 0 = 0$

If $i \neq l$ $\quad = \delta^i_l\,\varepsilon_{ilm} = 0$

(ii) ε_{iks} is non-zero only if i, k, s are all different.

ε_{mps} is non-zero only if m, p, s are all different.

Therefore either $i = m$ and $k = p$ or $i = p$ and $k = m$

If $i = m$ and $k = p$

$$\varepsilon_{iks}\varepsilon_{mps} = \varepsilon_{mps}\varepsilon_{mps} = 1$$

If $i = p$ and $k = m$

$$\varepsilon_{iks} = \varepsilon_{sik} = -\varepsilon_{ski} = -\varepsilon_{smp}$$

$$\varepsilon_{iks}\varepsilon_{mps} = -1$$

Now if i, m, k, p are kept fixed and s is given value 1, 2, 3 then only one term in the sum $\varepsilon_{iks}\varepsilon_{mps}$ is non-zero and hence the sum is either +1 or –1.

It can be seen from $\delta_{im}\delta_{kp} - \delta_{ip}\delta_{km} = +1$ for $i = m, k = p, i \neq k$

$= -1$ for $i = p, k = m, i \neq k$

Therefore,

$$\varepsilon_{iks}\varepsilon_{mps} = \delta_{im}\delta_{kp} - \delta_{ip}\delta_{km} = 0 \text{ except } \begin{cases} i = m, k = p, i \neq k \\ i = p, k = m, i \neq k \end{cases}$$

❑

Part-B : Newtonian Mechanics & Wave Motion

UNIT-V

Dynamics of a System of Particles

SECTION-A VERY SHORT ANSWER TYPE QUESTIONS

Q.1. Write about Johannes Kepler.

Ans. Johannes Kepler (1571-1630 AD), a German astronomer and mathematician, who was assistant of Tycho Brahe, made calculations from those measurements and described the orbit of planets around the Sun. He published three laws of planetary motion. We shall study these laws in a later chapter. Kepler's laws modified the heliocentric theory of Copernicus, replacing circular orbits of planets by elliptical orbits.

Q.2. What do you mean by coriolis force?

Ans. Coriolis Force : The invisible force that appears to deflect the wind is coriolis force. The coriolis force applies to movement on rotating objects. It is determined by the mass of the object and the objects rate of rotation. The coriolis force is perpendicular to the objects axis. The Earth spins on its from west to east.

Q.3. What do you understand by centripetal acceleration?

Ans. Centripetal acceleration : The acceleration of a body traversing a circular path. Because velocity is a vector quantity that is, it has both a magnitude, the speed and a direction. When a body travels on a circular path, it direction constantly change and thus its velocity changes producing an acceleration.

$$\text{Centripetal acceleration } (a_c) = \frac{v^2}{r}$$

where, a_c = centripetal acceleration, v = velocity, r = radius.

Q.4. Write about the Tycho Brahe Work.

Ans. Tycho Brahe Work : Tycho Brahe (1546-1601 AD) was a Danish astronomer. His work on astronomy consisted of measuring the positions of the stars, planets, Moon and the Sun. Every possible night and day he recorded these measurements year after year.

Q.5. What do you mean by conservation of linear momentum?

Ans. Conservation of Linear Momentum : This law of physics according to which the quantity called momentum that characterizes motion never changes is an isolated collection of objects, that is the total momentum of a system remains constant.

The linear momentum of a particle of mass m moving with a velocity $\vec{v}$ is given by

$$\vec{p} = m\vec{v}$$

It is a vector quantity and is directed along $\vec{v}$. Its MKS unit is kg-m/s and dimensions are $[MLT^{-1}]$. Its value depends upon the reference frame of the observer.

Q.6. What is Euler's principle?

Ans. Euler's first law states that the rate of change of linear momentum p of a rigid body is equal to the resultant of all the external forces F_{ext} acting on the body.
The linear momentum of a rigid body is the product of the mass of the body and the velocity of its centre of mass v_{cm}.

SECTION-B SHORT ANSWER TYPE QUESTIONS

Q.1. Write short note on Galileo discoveries.

Ans.

Galileo's Discoveries

Galileo Galilei (1564-1642 AD) was an Italian astronomer, philosopher, mathematician and physicist. His main discoveries may be summarized as :

1. **Support of Heliocentric Theory of Copernicus :** Galileo constructed a telescope and made pioneering observations of Universe. His observations supported the Copernicus theory (Sun centered Solar System). This brought him into serious conflict with the Church which forced him to make a public denial of his opinion and put him under restriction for later life.
2. **Galileo's Laws of Motion :** From his experiments on motion of bodies on inclined planes or freely falling bodies, he contradicted Aristotelian notion (that a force was required to keep a body in motion and speed in a medium depends upon weight). Galileo developed the concept of motion in terms of velocity (speed and direction). He developed the idea of force, as the cause of motion. He proposed that :
   (i) Natural state of an object is the rest or of uniform motion *i.e.,* objects always have velocity, sometimes magnitude of velocity is zero (= Rest).
   (ii) Objects resist change in motion. This property is called inertia. Hence a force is required to produce a change in motion.

Galileo's law of motion was adopted as the first law of motion by Newton. Newton brought all the threads of history of mechanics together and concluded that the laws which govern heavens are the same laws that govern motion on the earth.

Q.2. What is theory of impetus? Explain in brief.

Ans.

Theory of Impetus

This theory was put forward initially to reject Aristotle's explanation of projectile motion. The theory was introduced by John Philoponus in 6th century, elaborated by many physicists, finally established by **Jean Buridan** in 14th century.

Explanation of Projectile Motion : Projectile motion is a vibrant motion. Thus a force is required to keep the projectile moving. The question is: what makes it moving after it leaves the thrower? According to the theory of impetus :

1. The medium resists the motion. It does not aid (as suggested by Aristotle), and
2. After leaving the thrower, the projectile continues to move by an impetus.

"Impetus is a motive force implanted into moving body (projectile) by the mover (thrower)."

According to Buridan, impetus is a variable quantity whose force is determined by the speed and quantity of matter in moving body. Mathematically,

Impetus = Weight × Speed

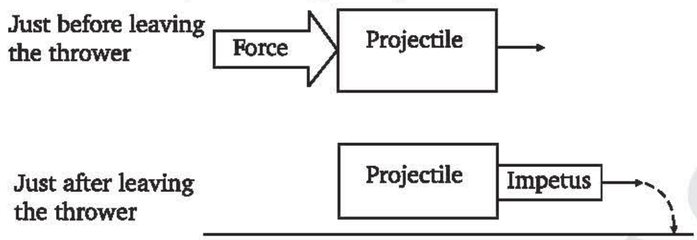


Fig. 1

As projectile moves, the resistance of medium applies force in a direction opposite to the motion; hence the impetus goes on diminishing, When it is almost destroyed, then gravitas of projectile prevails and moves the body to its natural place. According to this theory, impetus might also be the cause of eternal circular motions of celestial bodies.

Q.3. What are the limitations of Newton's laws?

Ans. **Limitations of Newton's Laws**

Newton's first law defines the force as the agent that produces change in motion and says that, if $\vec{F} = 0$ then we have $\vec{a} = 0$. This can happen only in inertial frames. Thus first law is valid only in inertial frames which is an obvious limitation.

The second law when mass m is constant, say that $\vec{F} = m\vec{a}$. Thus $\vec{F} = 0$ then $\vec{a}$ would also be zero. However in cases where the mass of the system does not remain constant, the second (and hence the first) law in the form $\vec{F} = m\vec{a}$ is not valid. Instead for variable mass systems such as falling raindrop or in the case of rocket we have to use the form $\vec{F} = \frac{d\vec{p}}{dt}$.

The third law requires the simultaneous measurement of the action and reaction forces, which is impossible. However, if the bodies interact for a sufficiently large time as compared to the time taken by the light signal to reach from one body to the other, then for all practical purposes, simultaneous measurement of the two forces is possible. For particles of **atomic dimensions**, Newton's third law does not hold good.

Another limitation of Newton's third law of motion is that it is not strictly correct when interaction between two bodies separated by a large distance is considered.

Q.4. What is Ferel's law? Explain.

Ans. **Coriolis effect on a Projectile**

Suppose the projectile is projected with a sufficiently large horizontal velocity, then coriolis force will act on the body for a sufficient period of time. This will make the position vector of the projectile to turn at a constant rate of $-\omega \sin \phi$. In the northern hemisphere, λ is positive. Hence the rotation as viewed from above is clockwise. Thus the projectile gets deflected towards the right (or east) in the northern hemisphere. But in the southern hemisphere, λ is negative and the rotation is counter-clockwise. Thus projectile gets deflected to the left or westwards. This is known as **Ferel's law**.

Q.5. The distance between the centres of the carbon and oxygen atoms in the CO molecule is 1.130×10^{-10} m. Locate the centre of mass of the molecule relative to the carbon atom.

Sol. The centre of mass of the CO molecule will lie on the line joining the C and O atoms. Let it be a distance x away from the C atom. Since atomic weights of C and O are respectively 12 and 16, therefore

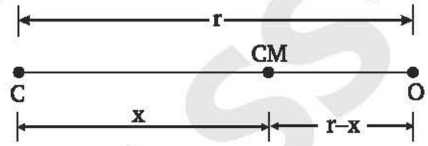


Fig.

$$m_C \cdot x = M_O \cdot (r - x)$$

or $$12x = 16(r - x)$$

or $$28x = 16r = 16\times1.130\times10^{-10}$$

$\therefore$ $$x = \frac{16\times1.130\times10^{-10}}{28}\text{ m} = 6.457\times10^{-11}\text{ m}$$

Thus the CM of the CO molecule lies at a distance of 6.457×10^{-11} m/s from C atom.

Q.6. A bullet of mass 20 g is fired with a speed of 1000 m/s from a freely hanging gun of mass 2.0 kg. Calculate the recoil velocity of gun.

Sol. Initial momentum of the system (gun + bullet) = 0

Final momentum of the system $= m_1v_1 + m_2v_2$

where m_1, m_2 are the masses of gun and the bullet respectively and v_1 and v_2 their final velocities.

According to the principle of conservation of linear momentum

final momentum = initial momentum

i.e., $$m_1v_1 + m_2v_2 = 0$$

or $$2.0\times v_1 + (20\times10^{-3})\times1000 = 0$$

or $$v_1 = \frac{-20}{2.0}\text{ m/s} = -10\text{ m/s}$$

The gun will therefore move with a velocity 10 m/s in a direction opposite to that of the bullet.

Q.7. Write short note on non-conservative force and conservation of total energy.

Ans. Non-conservative Force and Conservation of Total Energy

When non-conservative forces act on a system then mechanical energy does not remain conserved. This is because of the fact that work done by a non-conservative force in displacing a particle from a point to another point depends upon the actual path taken. Frictional and viscous forces are examples of non-conservative forces. Thus the work done in displacing the particle from *A* to *B* is not equal and opposite to that from *B* to *A*. As a result the total work done in the round trip is not zero. There is a loss of kinetic energy in moving from *A* to *B* and also from *B* to *A*. This loss of kinetic energy is equal to the work done during the round trip.

A part of kinetic energy is converted into some other forms of energy like sound, or heat or light energy etc. However, the sum total of all types of energies is always conserved. This is known as the general law of conservation of energy or conservation of total energy.

If both conservative and non-conservative forces are acting on a particle, then

$$W_{(C)} + W_{(N)} = \Delta K$$

or $$-\Delta U + W_{(N)} = \Delta K \qquad \because W_{(C)} = -\Delta U$$

or $$W_{(N)} = \Delta K + \Delta U = \Delta E$$

where $W_{(C)}$ and $W_{(N)}$ are respectively the work done by conservative and non-conservative forces and ΔK is the loss in kinetic energy of the particle. Thus the change in total energy is equal to the work done by non-conservative force. Particularly for a frictional force,

$$W_{(N)} = -Q$$

where Q is the heat developed. Then

$$\Delta E = -Q \quad \text{or} \quad \Delta E + Q = 0$$

This means that the change in total energy of the particle is zero *i.e.*, total energy remains conserved.

SECTION-C LONG ANSWER TYPE QUESTIONS

Q.1. Write contributions in the development of mechanics from Aristotelian Physics.

Ans.

Aristotelian Physics

Aristotle (384 BC-322 BC), a Greek philosopher and astronomer, proposed the abstract principles that govern the nature.

1. On Universe

For Aristotle, the earth is at the centre of the universe and is stationary. In other words, Universe is Geocentric.

Aristotle divided the Universe in two spherical regions known as Terrestrial sphere & Celestial sphere (fig. 1).

(i) **Terrestrial Sphere :** Terrestrial sphere is also known as sublunary region and consists of sphere of earth at the centre, surrounded by concentric shell of water, shell of air and finally the shell of fire [fig. 1(a)]. Terrestrial sphere is :

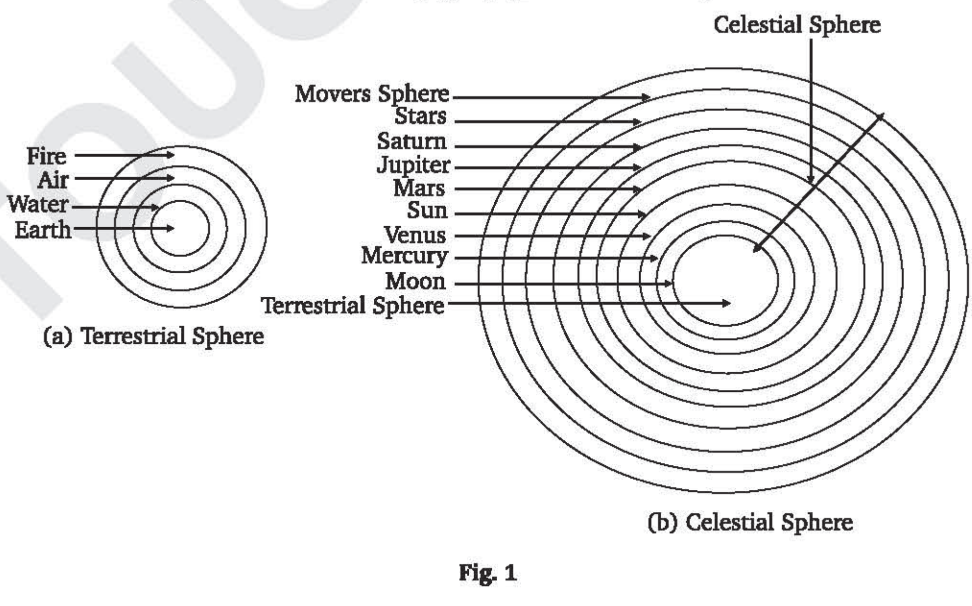


Fig. 1

- Inhabited by humans
- Corruptible
- Composed of four elements : earth, water, air and fire.

(ii) **Celestial Sphere :** Celestial sphere, also called superlunary region :

- Consists of heavenly bodies (Sun, Moon, Planets & Stars)
- Incorruptible
- Heavens are composed of special element called ether or quintessence (fifth element of Universe) which is pure, divine and incorruptible.

In celestial sphere the Sun, Moon, Planets- Mercury, Venus, Mars, Jupiter and Saturn (only 5 planets were known at that time) and stars exist in their respective shells [fig. 1(b)] which rotate at the fixed rate. In the outermost shell, there exists a prime mover of various shells which by spinning the star shell, imparts motion to other celestial shells.

2. On Motion of Bodies

Terrestrial Motion : Terrestrial bodies possess two types of motion natural motion and vibrant motion.

(i) **Natural Motion :** All terrestrial bodies move to their natural place (by themselves). Such a motion is called the natural motion.

In this motion, all bodies rise or fall towards their natural place in straight line path (straight up or straight down).

"Heavy objects fall down, very light objects rise up".

Elements	Natural Place
Earth	Centre of Earth
Water	Water shell around earth
Air	Air shell around water shell
Fire	Fire shell around air shell

Examples of Natural Motion : A stone falls down towards centre of the earth because stone is similar to the earth.

- Air bubble in a liquid rises up.
- Smoke rises up because smoke is similar to air.
- Objects fall (or rise) with speed (v) proportional to their weights (W) and inversely proportional to the density of fluid they are immersed in *i.e.,*

$$v \propto V$$

$$v \propto \frac{1}{e}$$

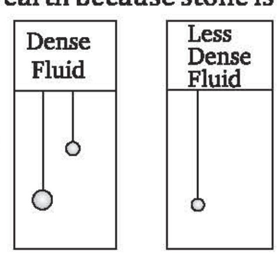


Fig. 2

These are the Aristotelian laws of motion. Aristotle believed that vacuum cannot exist because if it exists, speed of object in vacuum will be infinite, which is not possible.

(ii) **Vibrant or Unnatural Motion :** Forced motion of an object away from its natural place is called vibrant/unnatural motion. Vibrant motion :

- Imposed motion by applying external push or pull
- **Influenced by two factors :** Motive force F and resistance R of the medium. Thus speed of object in vibrant motion

$$v \propto \frac{F}{R} \, (F > R)$$

If $F \leq R$, no vibrant motion takes place.

Examples of Vibrant Motion :

(i) Motion of an arrow shot from a bow.

(ii) Motion of stone thrown at some angle etc.

How does an arrow shot from a bow continue to fly in air after it has left bow and string? Aristotle explained that arrow creating vacuum behind it into which air rushes and applies force to the back of arrow.

We can summarize Aristotle's views on motion as :

(i) **For terrestrial bodies :**

Natural state is the state of rest.

Force is required to keep the body moving. In natural motion, mover is gravitas (weight) *i.e.,* internal whereas in vibrant motion, mover is external.

(ii) **For Celestial bodies :** Natural state is the eternal circular motion (which is unchangeable).

Q.2. Define centre of mass of a system.

Ans.

Centre of Mass

For a many particle system, the translatory motion can be described in terms of the motion of representative point known as the centre of mass of the system.

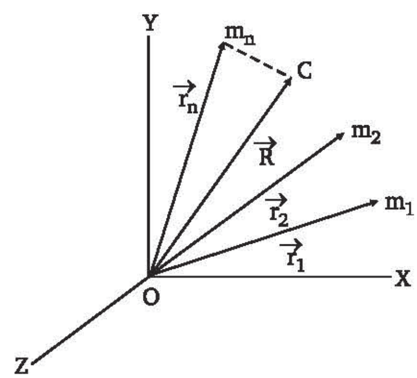


Fig.

The centre of mass of a system (or body) is defined as the representative point whose motion describes the actual motion of the system if all the forces act directly upon it and the entire mass of the system is concentrated at this point.

The concept of centre of mass is very useful when the body undergoes rotational or vibrational motion along with translational motion. Further it is very important in the analysis of collisions.

We consider a system of n particles of masses $m_1, m_2, ..., m_n$ whose position vectors relative to an arbitrary origin O are $\vec{r}_1, \vec{r}_2, ..., \vec{r}_n$ respectively (fig). The position vector of the centre of mass C of this system is defined as

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + ... + m_n \vec{r}_n}{m_1 + m_2 + ... + m_n}$$

or

$$\vec{R} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{M} \qquad ...(1)$$

where $M = m_1 + m_2 + ... + m_n$, is the total mass of the system.

Here $\sum_{i=1}^{n} m_i \vec{r_i}$ is called the first moment of mass for the system.

The centre of mass of homogeneous bodies of regular shapes lies at the point (or the line) of symmetry. For a sphere, the centre of mass lies at the centre of the sphere. For a cone, the centre of mass lies on the axis of symmetry. In such cases $\int \vec{r}\, dm = 0$ or $\vec{R} = 0$ *i.e.*, the centre of mass coincides with the geometric centre. The centre of mass does not necessarily lie within the body. It may also lie outside the body as in the case of a ring.

Particular Cases :

1. If the particles of the system all lie in a plane than the coordinates of the centre of mass (X, Y) are

$$X = \frac{1}{M}\int x\, dm; \; Y = \frac{1}{M}\int y\, dm$$

2. If all particles (n) lie on a straight line along x-axis, then the centre of mass also lie on that line having x-coordinate as

$$X = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n} = \frac{1}{M}\sum_i m_i x_i$$

It is important to note that the centre of mass of the system depends upon the masses of the particles and their relative positions.

Q.3. Describe Newton's laws of motion and show that the first law is contained in the second law.

Ans.

Newton's Laws of Motion

Sir Issac Newton gave three laws which govern motion of various objects. These are :

First Law (Law of Inertia)

"A body continues to remain in its state of rest or of uniform motion along a straight line so long as no external force acts on it".

The mass of the body in translational motion, is called its inertia. Thus different bodies possess different inertia. The property of any body to oppose the changes in its state of rest or of uniform motion is termed as inertia. Mathematically, first law may be stated as

$$\vec{a} = 0 \text{ if and only if } \vec{F} = 0$$

where $\vec{a}$ is the acceleration produced in the body and $\vec{F}$ is the resultant external force acting on it.

Second Law (Law of Force)

The time rate of change of linear momentum is proportional to the impressed (*i.e.*, external or applied) force and takes place in the direction of force *i.e.*

$$\vec{F} = \frac{d\vec{p}}{dt}$$

where $\vec{p} = m\vec{v}$, is the linear momentum of the body and $\vec{F}$ is the force applied on the body. If the mass m remains constant throughout the motion, then

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a}$$

where $\vec{a} = \frac{d\vec{v}}{dt}$ is the acceleration produced in the body. Here $\vec{a}$ and $\vec{F}$ should be measured at the same instant of time.

Third Law (Law of Action and Reaction)

To every action, there is always an equal and opposite reaction.

This law involves the interaction between two bodies. The force of action and reaction are **internal forces** and act upon two different bodies. Thus if F_{12} denote the force on body 1, due to body 2 and F_{21} that on body 2, due to body 1, then

$$F_{12} = -F_{21}$$

Now, $$F_{12} = m_1\frac{dv_1}{dt} \quad \text{and} \quad F_{21} = m_2\frac{dv_2}{dt}$$

∴ $$m_1\frac{dv_1}{dt} = -m_2\frac{dv_2}{dt}$$

or $$m_1a_1 = m_2a_2$$

where $$a_1 = \left|\frac{dv_1}{dt}\right| \quad \text{and} \quad a_2 = \left|\frac{dv_2}{dt}\right|$$

Thus $$m_2 = m_1\frac{a_1}{a_2}$$

Hence, Newton's third law defines mass uniquely.

Newton's First Law as *s* Special Case of Second Law

Newton's first law defines the force as an agent to cause a change in motion and says that

if $\vec{F} = 0$, then $\vec{a} = 0$

The second law says that the force is equal to the time rate of change of momentum *i.e.*,

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$$

where $\vec{p} = m\vec{v}$, is the momentum of the particle of mass m moving with a velocity $\vec{v}$. If the mass of the particle is constant, then

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

Thus if $\vec{F} = 0$, then again $\vec{a} = 0$, which is first law. Therefore, the first law is just a special case of the second law when $\vec{F} = 0$. Hence first law is contained in second law. The second law is, therefore, the real law of motion. Thus second law is the most general of all the three Newton's laws of motion.

Q.4. Discuss the effect of coriolis force on a freely falling object.

Ans.

Effect of Coriolis Force

Coriolis force, a type of fictitious force that acts on bodies in motion relative to rotating frame of reference of earth, produces certain interesting effects like cyclones (*e.g.*, hurricanes), trade winds and ocean currents etc.

We consider here two different cases :

(i) Coriolis effect on freely falling object.

(ii) Coriolis effect on a projectile.

But here only describe the coriolis effect on freely following object :

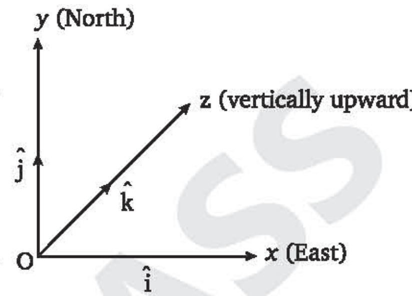


Fig. 1

Coriolis Effect on Freely Falling Object

The direction of coriolis force is not truly vertical. So we resolve it into horizontal and vertical components. The vertical component will affect the value of g whereas the horizontal component will produce horizontal deflection eastwards or westwards depending upon whether the object is in the northern or southern hemisphere respectively, as will be verified below. Considering axes and direction pattern according to given figure. If the falling object acquires a velocity $\vec{V}'$ at time t when it has fallen a vertical height of h, then

$$\vec{V}' = -v\hat{k} \qquad ...(1) \quad |\vec{V}'| = v.$$

If λ be the latitude of that place, then in the northern hemisphere

$$\vec{\omega} = (\omega\cos\lambda)\hat{j} + (\omega\sin\lambda)\hat{k} \qquad ...(2)$$

Coriolis acceleration, $\vec{a}_c = -2\vec{\omega}\times\vec{V}' = -2(\omega\cos\lambda\,\hat{j} + \omega\sin\lambda\,\hat{k})\times\{-v\hat{k}\}$

$$= 2\omega v\cos\lambda\,\hat{i} \qquad \because \hat{j}\times\hat{k} = i \text{ and } \hat{k}\times\hat{k} = 0$$

Thus coriolis acceleration on the falling object is along the east direction. Using Newton's second law,

$$\frac{d^2x}{dt^2} = 2\omega v\cos\lambda = 2\omega(gt)\cos\lambda\,\hat{i} \qquad \because v = u + gt = 0 + gt$$

On integration, we get the x-component of velocity of the particle as

$$V_x = \frac{dx}{dt} = \int(2\omega gt\cos\lambda)\,dt + C,$$

where C is the constant of integration. Thus

$$V_x = \omega gt^2\cos\lambda + C$$

But at $t = 0, V_x = 0 \quad \Rightarrow \quad C = 0$. Hence

$$V_x = \frac{dx}{dt} = \omega gt^2\cos\lambda$$

Integrating again, we get the displacement of the falling object along x-axis *i.e.*,

$$x = \int (\omega g \cos\lambda) t^2 dt + D$$

$$x = \left(\frac{1}{3}\omega g \cos\lambda\right) t^3 + D$$

again at $t = 0, x = 0 \quad \Rightarrow D = 0.$

Hence, $$x = \left(\frac{1}{3}\omega g \cos\lambda\right) t^3 \qquad ...(3)$$

To determine t, we have

$$h = ut + \frac{1}{2}gt^2 = 0 + \frac{1}{2}gt^2$$

or $$t = \sqrt{\frac{2h}{g}} \qquad ...(4)$$

Thus $$x = \left(\frac{1}{3}\omega g \cos\lambda\right)\left(\frac{2h}{g}\right)^{3/2}$$

$$x = \sqrt{\left(\frac{8}{9g}\right)} h^{3/2} \omega \cos\lambda \qquad ...(5)$$

Thus a freely falling object will be deflected towards right (or in the east direction) in the northern hemisphere by an amount given by eqn. (5). This deflection depends upon the value of latitude λ.

At the equator : $\lambda = 0$ so $x = \sqrt{\frac{8}{9g}} h^{3/2} \omega$

which is the maximum deflection.

At the poles : $\lambda = 90°$ so $\cos\lambda = 0$. Hence, $x = 0$. At the poles, there will be no deflection. The object falls at exactly a place vertically below the place from where it was dropped.

Now when the object falls vertically in the solution hemisphere, then

$$\vec{\omega} = -\omega\cos\lambda\,\hat{j} + \omega\sin\lambda\,\hat{k}$$

$$\therefore \quad \vec{a}_c = -2\,\vec{\omega} \times \vec{V'} = -2[-\omega\cos\lambda\,\hat{j} + \omega\sin\lambda\,\hat{k}] \times (-v\hat{k})$$

$$= -2\,\omega v \cos\lambda i$$

Thus coriolis acceleration is along negative direction of x-axis (*i.e.*, along west). Hence coriolis effect will deflect the freely falling object towards the left (or west) in the southern hemisphere.

Q.5. Discuss about a frame of reference rotating with respect to an inertial frame and explain various pseudo forces.

Ans.

Rotating Frame of Reference

Let S be an inertial frame of reference with coordinate axes designated as X, Y and Z with its origin at O (Fig. 1). Consider another frame S' having coordinate axes X', Y' and Z' and origin coinciding with that of frame S. Let this frame S' is rotating counter clockwise relative to frame

S with an angular velocity ω (not necessarily constant) and the axis of rotation pass through the common origin O of the two frames of reference. A common example of a rotating reference frame is the surface of earth. The point P represents the position of any moving point object at time t. Its position vector $\vec{r}$ at any time t will be the same relative to frames S and S'. But the components of $\vec{r}$ will be different along the three axes in these frames.

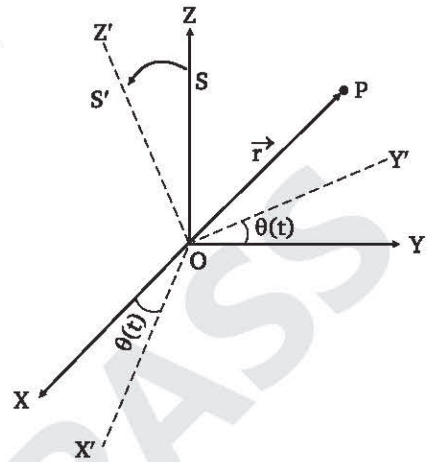


Fig. 1

Let $\hat{i}, \hat{j}, \hat{k}$ be the unit vectors along axes X', Y', Z' in the rotating frame of reference. Let these vectors rotate with an angular speed $\omega(t)$ about an axis $\vec{\omega}(t)$, then for any general unit vector $\hat{u}$ in the rotating frame, it can be shown that

$$\frac{d\hat{u}}{dt} = \vec{\omega}(t) \times \hat{u} \qquad \text{...(1)}$$

The time derivative for a vector

$$\vec{F}(t) = \hat{i}\,F_x(t) + \hat{j}\,F_y(t) + \hat{k}\,F_z(t)$$

becomes

$$\frac{d\vec{F}(t)}{dt} = \left(\frac{dF_x}{dt}\hat{i} + \frac{d\hat{i}}{dt}F_x\right) + \left(\frac{dF_y}{dt}\hat{j} + \frac{d\hat{j}}{dt}F_y\right) + \left(\frac{dF_z}{dt}\hat{k} + \frac{d\hat{k}}{dt}F_z\right)$$

because unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$ are not constant vectors. Thus

$$\frac{d\vec{F}(t)}{dt} = \left(\frac{dF_x}{dt}\hat{i} + \frac{dF_y}{dt}\hat{j} + \frac{dF_z}{dt}\hat{k}\right) + [\vec{\omega}(t) + (\hat{i}\,F_x + \hat{j}\,F_y + \hat{k}\,F_z)]$$

on using eqn. (1)

$$\frac{d\vec{F}(t)}{dt} = \left(\frac{d\vec{F}}{dt}\right) + \vec{\omega}(t) + \vec{F}(t) \qquad \text{...(2)}$$

where $\left(\frac{d\vec{F}}{dt}\right)_r$ represents the time rate of change of $\vec{F}$ as observed in the rotating frame.

The equation (2) may be written as an operator equation

$$\frac{d}{dt}\vec{F} = \left[\left(\frac{d}{dt}\right)_r + \vec{\omega}(t)\right] \times \vec{F} \qquad \text{...(3)}$$

and can be applied to any time dependent vector function $\vec{F}$ like position vector, velocity etc. Equation (3) is also called transport theorem. The **LHS** of this equation gives time derivative of any vector function in the inertial frame (S) and **RHS** gives time derivative of same vector function in the rotating frame and consists of sum of two terms. The first term is due to the explicit time dependence of this function and the second term arises due to the rotation of the frame.

Let this operator equation (3) be applied to the position vector $\vec{r}$ of the particle in the frames S and S' *i.e.*, $\vec{F} = \vec{r}$, then

$$\left(\frac{d\vec{r}}{dt}\right)_S = \left(\frac{d\vec{r}}{dt}\right)_{S'} + \vec{\omega}(t)\times\vec{r} \qquad ...(4)$$

or $$\vec{V} = \vec{V}' + \vec{\omega}(t)\times\vec{r} \qquad ...(5)$$

where $\vec{V} = \left(\frac{d\vec{r}}{dt}\right)_S$ and $\vec{V}' = \left(\frac{d\vec{r}}{dt}\right)_{S'}$ respectively.

Now apply the same operator equation (3) for the velocity $\vec{V}$ of the particle *i.e.*, $\vec{F}(t) = \vec{V}(t)$, then

$$\left(\frac{d\vec{V}}{dt}\right)_S = \left(\frac{d\vec{V}}{dt}\right)_{S'} + \vec{\omega}(t)\times\vec{V}$$

$$= \left[\frac{d}{dt}(\vec{V}' + \vec{\omega}(t)\times\vec{r})\right] + \vec{\omega}(t)\times(\vec{V}' + \vec{\omega}(t)\times\vec{r})$$

on using equation (5)

$$= \frac{d\vec{V}'}{dt} + \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}(t)\times\left(\frac{d\vec{r}}{dt}\right)_{S'} + \vec{\omega}(t)\times\vec{V}' + \vec{\omega}(t)\times(\vec{\omega}(t)\times\vec{r})$$

or $$\vec{a} = \vec{a}' + \frac{d\vec{\omega}}{dt}\times\vec{r} + 2\vec{\omega}\times\vec{V}' + \vec{\omega}\times(\vec{\omega}\times\vec{r}) \qquad ...(6)$$

Here $\vec{a} = \frac{d\vec{V}}{dt}$ and $\vec{a}' = \frac{d\vec{V}'}{dt}$ represent the acceleration of the particle in frame S and S' respectively. Remember $\vec{\omega}$ is a time dependent parameter through we have omitted explicit time dependence notation. Thus the acceleration of the particle in the rotating frame is

$$\vec{a}' = \vec{a} - 2\vec{\omega}\times\vec{V}' - \vec{\omega}\times(\vec{\omega}\times\vec{r}) - \frac{d\vec{\omega}}{dt}\times\vec{r} \qquad ...(7)$$

which consists of three additional terms. The second term of RHS *i.e.*, $-2\vec{\omega}\times\vec{V'}$ is called the **Coriolis acceleration** and the third term *i.e.*, $-\vec{\omega}\times(\vec{\omega}\times\vec{r})$ is called the **centrifugal acceleration**. The last term *i.e.*, $-\frac{d\vec{\omega}}{dt}\times\vec{r}$ which is non-zero only when the frame S' is in non-uniform rotation, called the **Euler acceleration**. All these accelerations are called fictitious accelerations and must be added to the true acceleration $(\vec{a})$ of the particle in frame S, to give the observed acceleration (a') of the particle in frame S'.

For a uniformly rotating frame, $\frac{d\vec{\omega}}{dt}=0$. Hence Euler acceleration will become zero in such a frame of reference.

If m be the mass of the particle P, then $\vec{F}=m\vec{a}$ = force on the particle in frame S, and $\vec{F'}=m\vec{a'}$ = force on the particle in frame S'.

Therefore, from eqn. (7)

$$m\vec{a'}=m\vec{a}-2m\vec{\omega}\times\vec{V'}-m\vec{\omega}\times(\vec{\omega}\times\vec{r})-m\frac{d\vec{\omega}}{dt}\times\vec{r}$$

or $$\vec{F'}=\vec{F}+\vec{F_0} \qquad ...(8)$$

where $$\vec{F_0}=-2m\vec{\omega}\times\vec{V'}-m\vec{\omega}\times(\vec{\omega}\times\vec{r})-m\frac{d\vec{\omega}}{dt}\times\vec{r}$$

is called the fictitious force which appears to be acting on the particle in frame S'. This force $\vec{F_0}$ is a consequence of the rotation of the frame S' itself and must be added to the true force $\vec{F}$ acting on the particle to get the observed (or measured) force $\vec{F'}$ on the particle in frame S'. In a non-rotating reference frame total fictitious force will be zero. Here

$$-2m\vec{\omega}\times\vec{V'} = \text{coriolis force}$$

$$-m\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \text{centrifugal force, and}$$

$$-m\frac{d\vec{\omega}}{dt}\times\vec{r} = \text{Euler force}$$

Thus total fictitious force $\vec{F_0}$ is the sum of coriolis force, centrifugal force and the Euler force. It must be noted that :

(i) Coriolis force acts on a particle which is moving in a frame of reference that is itself rotating about an inertial frame of reference. This force will be zero when (a) either the particle is at rest in the rotating frame of reference, or (b) $\vec{V'}$ is parallel to $\vec{\omega}$. It is proportional to both $\vec{\omega}$ and $\vec{V}'$ and is always directed perpendicular to $\vec{V}'$. If $\vec{\omega}$ is in the counter clock-wise direction, then direction of coriolis force will be obtained by a 90° rotation relative to $\vec{V}'$ in the clock-wise direction as shown in figure 2. Thus for counter-clockwise rotation of frame, coriolis force acts towards the right of the motion of the particle whereas for clockwise rotation, it acts towards the left of the particle motion.

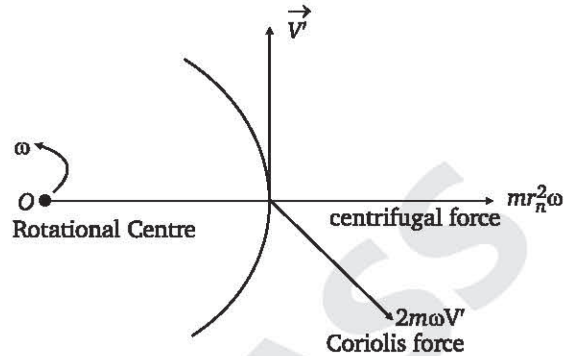


Fig. 2

(ii) Centrifugal force acts on a particle which is at rest relative to a rotating frame of reference and is numerically equal to the centripetal force but in the opposite direction. Its magnitude is $mr_n^2\omega$, where r_n is the perpendicular distance between the particle and the axis of rotation. Being position dependent, it is a conservative force. It is the only fictitious force acting on a particle at rest relative to a uniformly rotating reference frame.

(iii) In the special case of uniformly rotating frame of reference, Euler force will be zero and then only coriolis force and centrifugal forces will constitute the total fictitious force.

Q.6. What is the effect of centrifugal force on acceleration due to gravity. Show that

$$g\lambda = g - \omega^2 R\cos^2\lambda$$

Ans.

Effect of Centrifugal Force

We consider a particle P at rest in the rotating reference frame of the earth. Only fictitious force acting on the particle will be the centrifugal force as shown in the figure. Here true direction of acceleration of the particle g is along PO and the observed or apparent direction of acceleration g_λ is along PN, such that

$$\vec{g}_\lambda = \vec{g} - \vec{\omega} \times (\vec{\omega} \times \vec{R}_n)$$

$$\vec{g} = -g(\cos\lambda\,\hat{i} + \sin\lambda\,\hat{j})$$

$$\vec{\omega} = \omega\hat{j}$$

and $\quad R_n = R\cos\lambda\,\hat{i}$

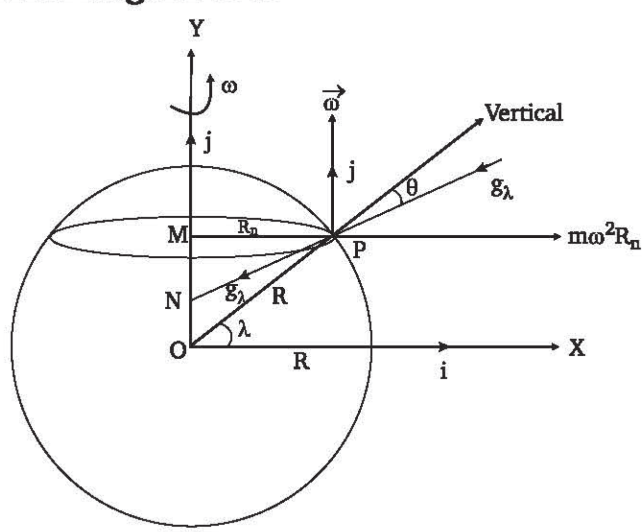


Fig.

where $\hat{i}$ and $\hat{j}$ are unit vectors along OX and OY directions *i.e.*, perpendicular and along the $\vec{\omega}$ respectively. Thus

$$\vec{g_\lambda} = -g\,(\cos\lambda\,\hat{i} + \sin\lambda\,\hat{j}) - \omega\,\hat{j} \times (\omega\,\hat{j} \times R\cos\lambda\,\hat{i})$$

$$= -g\,(\cos\lambda i + \sin\lambda j) - \omega\,\hat{j} \times (-\omega R\cos\lambda\,\hat{k})$$

$$= -g\,(\cos\lambda\,\hat{i} + \sin\lambda\,\hat{j}) + \omega^2 R\cos\lambda\,\hat{i}$$

$$\vec{g_\lambda} = -(g - \omega^2 R)\cos\lambda\,\hat{i} - (g\sin\lambda)\,\hat{j}$$

$\therefore$ Magnitude of apparent acceleration $\vec{g_\lambda}$ is

$$|\vec{g_\lambda}| = g_\lambda = \sqrt{\{-(g - \omega^2 R)\cos\lambda)\}^2 + \{-g\sin\lambda\}^2}$$

$$= \sqrt{g^2\cos^2\lambda + \omega^4 R^2\cos^2\lambda - 2g\omega^2 R\cos^2\lambda + g^2\sin^2\lambda}$$

$$= \sqrt{g^2 - 2g\omega^2 R\cos^2\lambda}$$

neglecting the term containing ω^4, as ω is small

or
$$g_\lambda = g\left(1 - \frac{2\omega^2 R}{g}\cos^2\lambda\right)^{1/2}$$

$$\approx g\left(1 - \frac{1}{2} \times \frac{2\omega^2 R}{g}\cos^2\lambda\right) \text{ using binomial theorem}$$

$$g_\lambda = g\left(1 - \frac{\omega^2 R}{g}\cos^2\lambda\right) \qquad ...(1)$$

If the apparent acceleration direction (PN) makes an angle θ with the direction (PO) of the true acceleration g, then

$$\tan\theta = \frac{g\cos\lambda - \omega^2 R\cos\lambda}{g\sin\lambda}$$

or
$$\theta = \tan^{-1}\left[\left(1 - \frac{\omega^2 R}{g}\right)\cot\lambda\right] \qquad ...(2)$$

Thus the centrifugal force not only reduces the effective value of g on the earth's surface but also slightly tilts its direction towards the north or south in the northern and southern hemispheres respectively. Both of these magnitude and direction changes depend upon the latitude λ of the place.

At the equator : $\lambda = 0$, $\therefore \cos\lambda = 1$ and $g_\lambda = g - \omega^2 R$ (maximum)

At the poles : $\lambda = 90°$, $\therefore \cos 90° = 0$ and $g_\lambda = g$ (maximum)

Thus the minimum value of g at the equator favours launching of satellites from the equatorial regions.

Q.7. Discuss kinetic and potential energy and also discuss about conservation of mechanical energy.

Ans.

Kinetic Energy

The kinetic energy K (or T) of a particle (or body) is its ability to do work by virtue of its motion. We consider the motion of a body of mass m under the action of a force $\vec{F}$. Let the body moves from position $\vec{r_1}$ to $\vec{r_2}$ and its velocity increases from $\vec{v_1}$ to $\vec{v_2}$. Then work done by the force on the body is given by

$$W = \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} \qquad ...(1)$$

where $d\vec{r}$ is an infinitesimal displacement of the body during which force may be assumed to be constant. If $\vec{a}$ be the acceleration of the body, then from Newton's second law

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d\vec{v}}{d\vec{r}} \cdot \frac{d\vec{r}}{dt}$$

or

$$\vec{F} = m\vec{a} = m\vec{v}\frac{d\vec{v}}{d\vec{r}} \qquad \because \frac{d\vec{v}}{dt} = \vec{v}$$

putting this value of $\vec{F}$ in equation (1),

$$W = \int_{r_1}^{r_2} \left(m\vec{v}\frac{d\vec{v}}{d\vec{r}} \right) \cdot d\vec{r} = m\int_{v_1}^{v_2} \vec{v} \cdot d\vec{v} \qquad ...(2)$$

Now $\qquad \vec{v}^2 = v^2$ or $\vec{v} \cdot \vec{v} = v^2$

On differentiating, we get $\vec{v} \cdot d\vec{v} = v dv$, therefore (2) becomes

$$W = m\int_{v_1}^{v_2} v dv = m\left(\frac{v^2}{2}\right)_{v_1}^{v_2}$$

or

$$W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \qquad ...(3)$$

The quantity $\frac{1}{2}mv^2$ is defined as the kinetic energy K of the body.

Thus

$$W = K_2 - K_1 = \Delta K \qquad ...(4)$$

where $K_1 = \frac{1}{2}mv_1^2$ and $K_2 = \frac{1}{2}mv_2^2$ are respectively the initial and final state kinetic energies of the body. Equation (4) says that, "the work done by the force on the body is equal to the increase in kinetic energy of the body". This principle is called work-energy theorem or work-energy principle.

If $K_2 = 0$, then $K_1 = -W$

i.e., $K_1 = \frac{1}{2}mv_1^2$ is the amount of work that a body can do before it comes to rest starting from an initial velocity v_1.

If the work is done by the force then body gains in kinetic energy and if work is done by the body then it loses some kinetic energy. In each case amount of work done is equal to the change in kinetic energy of the body. Work-energy principle is valid even in the presence of **non-conservative forces.**

Potential Energy

The potential energy of a particle or a body is defined as its capacity to do work due to its position or configuration. For example a compressed spring has some energy stored in it in the form of potential energy. This spring is capable of doing some work at the cost of its potential energy. Similarly, the water stored in a dam or reservoir has the ability to run the turbine. So we can say that potential energy of a body is a form of stored Energy that can be converted fully into kinetic energy. It is usually denoted by U or V.

The potential energy can be defined only for conservative force fields (for which work done between two points is independent of the path between those points). Let in a conservative force field, the present position of the body is represented by $\vec{r}$ and its reference or standard position by $\vec{r_0}$ (in which $U = 0$).

$$U = \int_r^{r_0} \vec{F} \cdot d\vec{r} = -\int_{r_0}^{r} \vec{F} \cdot d\vec{r} \quad \text{... (1)}$$

The potential energy is therefore the amount of work done by the force to restore the body from its present position to the standard position. In the standard position the capacity of body to do work becomes zero and the force acting on the particle or body is also zero. For a spring, the unstretched or normal state can be taken as the standard condition.

We may note that :

1. For a non-conservative force, the work done cannot be expressed in the form of potential energy because it depends upon the direction of motion as well as on the path taken. In some cases the work done may also depend upon the speed of the body. Thus potential energy cannot be defined for non-conservative forces.
2. The standard position is sometimes taken at $\vec{r_0} = \infty$.

$$U = -\int_{\infty}^{r} \vec{F} \cdot d\vec{r} \quad \text{...(2)}$$

   So the potential energy is equal to the work done in moving the body from infinity to present position $\vec{r}$ against the force.
3. If the particle is displaced along the direction of force, then

$$U = -\int_{r_0}^{r} \vec{F} \cdot d\vec{r} = -\int_{r_0}^{r} F\,dr\cos 0° = -\int_{r_0}^{r} F \cdot dr$$

   On differentiating w.r.t. r, we get

$$F = -\frac{dU}{dr}$$

So, the potential energy is a function of position whose negative gradient gives the force acting on the body.

4. Potential energy in the initial position i, using equation (1) is

$$U_i = \int_{r_i}^{r_0} \vec{F} \cdot d\vec{r}$$

and in the final position f, $\quad U_f = \int_{r_f}^{r_0} \vec{F} \cdot d\vec{r}$

The change in potential energy in moving the body from i to f is

$$U_f - U_i = \int_{r_f}^{r_0} \vec{F} \cdot d\vec{r} - \int_{r_i}^{r_0} \vec{F} \cdot d\vec{r} = \int_{r_f}^{r_0} \vec{F} \cdot d\vec{r} + \int_{r_0}^{r_i} \vec{F} \cdot d\vec{r} = \int_{r_f}^{r_i} \vec{F} \cdot d\vec{r}$$

or $\quad U_f - U_i = \Delta U = -\int_{r_i}^{r_f} \vec{F} \cdot d\vec{r} = -W$

Thus increase in potential energy of the body is equal to negative of the work done by the force in moving the body from initial to final position.

Law of Conservation of Mechanical Energy

The sum of kinetic and potential energies of a body is called its mechanical energy.

We know that the work done in displacing a particle from A to B under the action of a conservation force $\vec{F}$, is

$$W = \int_A^B \vec{F} \cdot d\vec{r} = \int_A^B (-dU) = U_A - U_B \qquad \ldots(1)$$

because $\quad \vec{F} \cdot d\vec{r} = (iF_x + jF_y + kF_z) \cdot (idx + jdy + kdz)$

$$= F_x dx + F_y dy + F_z dz = -\frac{\partial U}{\partial x} dx - \frac{\partial U}{\partial y} dy - \frac{\partial U}{\partial z} dz$$

or $\quad \vec{F} \cdot d\vec{r} = -dU$

We also have $\quad W = \int_A^B \vec{F} \cdot d\vec{r} = \int_A^B m\vec{v} \frac{d\vec{v}}{d\vec{r}} \cdot d\vec{r} = \int_A^B m\vec{v} \cdot d\vec{v}$

or $\quad W = m\int_A^B v\, dv = \frac{1}{2} mv_B^2 - \frac{1}{2} mv_A^2 = K_B - K_A \qquad \ldots(2)$

where $K_A = \frac{1}{2} mv_A^2$ and $K_B = \frac{1}{2} mv_B^2$ are the potential energies in the position A and B

Equating (1) and (2), we get

$$U_A - U_B = K_B - K_A \quad \text{or} \quad K_A + U_A = K_B + U_B$$

or $\quad K + U = E$, a constant.

Here E, is called the mechanical energy of the particle or body. Thus mechanical energy of a body remains conserved under the action of a conservative force. This is known as the law of conservation of mechanical energy.
It should be noted that the mechanical energy of the body remains conserved with respect to both the position of the body and the time. This means that in a conservative field force, the value of mechanical energy E does not change either due to change in the position of the body or due to the passage of time.

Q.8. The position of a moving particle is at any instant given by $\vec{r} = A\cos\theta i + A\sin\theta j$. Show that the force acting on it is a conservative one. Also determine the potential function.

Sol. The position vector of the particle at any instant t is

$$\vec{r} = A\cos\theta i + A\sin\theta j = A\cos\omega t i + A\sin\omega t j$$

where $\theta = \omega t$, and ω is the angular velocity of the particle. Linear velocity of the particle at the given instant is

$$\vec{V} = \frac{d\vec{r}}{dt} = A\omega(-\sin\omega t i + \cos\omega t j)$$

Its acceleration $$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -A\omega^2 (\cos\omega t i + \sin\omega t j) = -\omega^2\vec{r}$$

If m be the mass of the particle, then force acting on it is

$$\vec{F} = m\vec{a} = -m\omega^2\vec{r} = -m\omega^2(xi + yj + zk)$$

This force will be conservative if $\vec{\nabla} \times \vec{F} = 0$

Now, $$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -m\omega^2 x & -m\omega^2 y & -m\omega^2 z \end{vmatrix}$$

$$= -m\omega^2\left[i\left(\frac{\partial}{\partial y}z - \frac{\partial y}{\partial z}\right) + j\left(\frac{\partial}{\partial z}x - \frac{\partial}{\partial x}z\right) + k\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right)\right]$$

$$= -m\omega^2[i(0-0) + j(0-0) + k(0-0)]$$

$\therefore$ $\vec{\nabla} \times \vec{F} = 0$. Hence force acting on the particle is conservative.

The corresponding potential function is given by

$$U = -\int \vec{F} \cdot d\vec{r} = -\int(-m\omega^2\vec{r}) \cdot d\vec{r} = m\omega^2\int \vec{r} \cdot d\vec{r} = m\omega^2\int r \cdot dr$$

$$U = \frac{1}{2}m\omega^2 r^2$$

Q.9. Describe motion of the centre of mass in detail.

Ans.

Motion of the Centre of Mass

Let us consider the motion of a system of n-particles of masses $m_1, m_2, \ldots, m_n$ having position vectors $\vec{r_1}, \vec{r_2}, \ldots, \vec{r_n}$ respectively. The position vector of the centre of mass of the system is defined as

$$\vec{R} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2} + \ldots + m_n \vec{r_n}}{m_1 + m_2 + \ldots + m_n} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r_i}$$

or
$$M \vec{R} = \sum_{i=1}^{n} m_i \vec{r_i} \quad \ldots(1)$$

where $M = m_1 + m_2 + \ldots + m_n$ is the total mass of the system which is taken to be constant. Differentiating w.r.t. time t, we get

$$M \frac{d\vec{R}}{dt} = \sum_{i=1}^{n} m_i \frac{d\vec{r_i}}{dt} = m_1 \frac{d\vec{r_1}}{dt} + m_2 \frac{d\vec{r_2}}{dt} + \ldots + m_n \frac{d\vec{r_n}}{dt}$$

or
$$M \vec{V} = m_1 \vec{V_1} + m_2 \vec{V_2} + \ldots + m_n \vec{V_n} = \sum_{i=1}^{n} m_i \vec{V_i} = \vec{P} \quad \ldots(2)$$

where $\vec{V} = \frac{d\vec{R}}{dt}$ is the velocity of the centre of mass and $\vec{V_1}, \vec{V_2}, \ldots, \vec{V_n}$ are the velocities of individual particles such that $\vec{V_1} = \frac{d\vec{r_1}}{dt}$ etc. Equation (2) says that the total linear momentum $\vec{P}$ of the system is equal to the product of the total mass of the system and the velocity of the centre of mass.

The velocity of the centre of mass is
$$\vec{V} = \frac{\sum_{i=1}^{n} m_i \vec{V_i}}{M} \quad \ldots(3)$$

Since in the absence of any external force, the momentum of the system remains conserved, therefore from equation (2)

$$M\vec{V} = \vec{P} = \text{Constant} \quad \text{or} \quad \vec{V} = \text{Constant} \quad \ldots(4)$$

Thus in the absence of external forces the centre of mass of a system moves with a constant velocity.

An example that verifies this fact is the case of decaying radioactive nucleus in flight. the particles of decay may move with different velocities in different directions but their centre of mass continues to move with the same velocity in the original direction.

Now differentiating equation (2) w.r.t. time t,

$$M \frac{d\vec{V}}{dt} = m_1 \frac{d\vec{V_1}}{dt} + m_2 \frac{d\vec{V_2}}{dt} + \ldots + m_n \frac{d\vec{V_n}}{dt}$$

or
$$M \vec{a}_{cm} = m_1 \vec{a_1} + m_2 \vec{a_2} + \ldots + m_n \vec{a_n}$$

where $\vec{a}_{cm} = \frac{d\vec{V}}{dt}$ is the acceleration of the centre of mass and $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are the accelerations of individual particles of the system.

From Newton's second law of motion $m_1 \vec{a}_1 = \vec{F}_1, m_2 \vec{a}_2 = \vec{F}_2, \ldots m_n \vec{a}_n = \vec{F}_n$ are the forces acting on different particles. Then we have

$$M\vec{a}_{cm} = \vec{F}_1 = \vec{F}_2 + \ldots + \vec{F}_n \qquad \ldots(5)$$

The force on RHS of (5) also include internal forces exerted by particles on each other. But from Newton's third law the vector sum of all internal forces will be zero as they occur in pairs of equal and opposite forces. Thus, internal forces play no role in the motion of centre of mass. Hence only the external forces govern the motion of the centre of mass. Therefore

$$M\vec{a}_{cm} = \vec{F}^{ext} \qquad \ldots(6)$$

where $\vec{F}^{ext}$ is the total external force acting on the system. Equation (6) says that the centre of mass of the system is concentrated at this point and the net external force acting on the system is applied directly to this point.

Q.10.1 kg, 2 kg and 3 kg masses are placed at three corners of equilateral triangle having each arm 1 meter, calculate the centre of mass of this system.

Sol. Let us consider the system of particles as shown in given fig.

Since $m_1 = 1$ kg lies at the origin, its position vector relative to origin $O, \vec{r}_1 = 0$.

$$\text{p.v. of mass } m_2 \ (=2\text{ kg}), \vec{r}_2 = 1i$$

$$\text{p.v. of mass } m_3 \ (=3\text{ kg}), \vec{r}_3 = i\cos 60° + j\sin 60° = \frac{1}{2}i + \frac{\sqrt{3}}{2}j$$

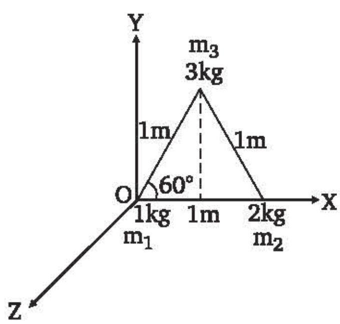
where i and j are unit vectors along X and Y axis respectively.

The position vector (p.v.) of the centre of mass relative to origin O(*i.e.*, particle of mass 1 kg) is

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3} = \frac{1\times 0 + 2\times 1i + 3\times\left(\frac{1}{2}i + \frac{\sqrt{3}}{2}j\right)}{1+2+3}\text{ meter}$$

$$= \frac{0 + 2i + \frac{3}{2}i + \frac{3\sqrt{3}}{2}j}{6}\text{ meter}$$

$$\vec{R} = \left(\frac{7}{12}i + \frac{\sqrt{3}}{4}j\right)\text{ meter}$$



Therefore the coordinates of the *COM* are $\left(\frac{7}{12}, \frac{\sqrt{3}}{4}, 0\right)$.

❑

UNIT-VI

Dynamics of a Rigid Body

SECTION-A VERY SHORT ANSWER TYPE QUESTIONS

Q.1. What is rotational motion?

Ans. Rotational motion can be defined as a motion of an object around a circular path, in a fixed orbit. It can also be defined as the motion of a body in which all of its particles move in a circular motion with a common angular velocity, about a fixed point, for example the rotation of earth about its axis.

Q.2. What is moment of inertia?

Ans. Moment of inertia in physics quantitative measure of the rotational inertia of a body *i.e.*, the opposition that the body exhibits to having its speed of rotation about an axis altered by the application of a torque (turing force).

Q.3. How is the radius of gyration defined for a regular solid sphere?

Ans. The radius of gyration is the square root of the average squared distance of a sphere object from the midpoint of the mass of the body.

Q.4. What are the significance of radius of gyration?

Ans. Following are the significant points :

(i) The radius of gyration is hedge in the figuring of the catching heap of a beam or pressure.

(ii) It is additionally valuable in the appropriation of intensity between the cross area of a given section.

(iii) The range of gyration is useful in contrast with the exhibition of various types of primary shapes at the hour of the pressure.

(iv) Lesser estimation of the radius of gyration is successful in performing the primary investigation.

(v) Lesser estimation of the radius of gyration shows that the rotational axis at which the segment catches.

Q.5. What is the effect of force on : (i) rigid body and (ii) non-rigid body.

Ans. (i) Applied force can not change the shape of the body but it can change its position.

(ii) Applied force can change shape as well as position of body.

Q.6. When a whole particle of a body move at the same distance which motion is called.

Ans. Rigid body motion.

Q.7. Does a rotating object have kinetic energy?

Ans. A rotating object also has kinetic energy. When an object is rotating about its centre of mass. Its rotational kinetic energy is $k=\frac{1}{2}/w^2$. Rotational kinetic energy $=\frac{1}{2}$ moment of inertiax (angular speed)^2.

Q.8. What is meant by stress?

Ans. Stress is a quantity that describe the magnitude of forces that cause deformation. Stress is generally defined as force per unit area. When forces pull on an object and cause its elongation, like the stretching of an elastic band we call such stress a tensile stress.

Q.9. Define strain.

Ans. Strain is simply the measure of how much an object is stretched or deformed. Strain occurs when force is applied to an object. Strain deals mostly with the change in length of the object.

Q.10. Define elastic constant. What is elastic constant formula?

Ans. Elastic constants are the parameters expressing the relation between the stress and the strain on the materials within the stress range that the materials exhibit elastic behaviour.

$$\text{Elastic constant } E = 9\ KGG + 3\ K$$

where, K is the bulk modulus, G is shear modulus of rigidity, E is Young's modulus of elasticity.

Q.11. What do you mean by cantilever?

Ans. Cantilever, beam supported at one end and carrying a load at the other end or distributed along the unsupported portion. Cantilever are employed extensively in building construction and in machine. In building, any beam built into a wall and with the free and projecting forms a cantilever.

Q.12. Find the radius of gyration for a solid sphere about diameter whose radius is $\sqrt{5}$ m.

Sol. If M be the mass and R the radius of the solid sphere, then its MI about a diameter is

$$I = \frac{2}{5} MR^2$$

If the radius of gyration of the sphere about a diameter is K, then

$$I = MK^2$$

So, $$MK^2 = \frac{2}{5} MR^2$$

⇒ $$K = \sqrt{\frac{2}{5}} R$$

or $$K = \sqrt{\frac{2}{5}} \times \sqrt{5}$$

$$= \sqrt{2} = 1.41 \text{ meter}$$

SECTION-B SHORT ANSWER TYPE QUESTIONS

Q.1. Explain the principle of conservation of angular momentum. Prove that

$$\vec{\tau} = \frac{d\vec{J}}{dt}.$$

Ans.

Law of Conservation of Angular Momentum

When body is rotating some fixed axis under the action of external torque $\vec{\tau}_{ext}$,, then the time rate of change of its angular momentum $\vec{J}$ is equal to the torque *i.e.*,

$$\frac{d\vec{J}}{dt} = \vec{\tau}_{ext}$$

If $\quad \vec{\tau}_{ext} = 0$, then $\frac{d\vec{J}}{dt} = 0 \quad$ or $\quad \vec{J} = \text{constant}$

This is the principle of conservation of angular momentum. According to it, "if the total external torque acting on a body is zero, then the angular momentum of the body remains conserved".

If the angular momentum of various particles of the system be $\vec{J}_1, \vec{J}_2, \ldots, \vec{J}_n$ then according to principle of conservation of angular momentum, in the absence of external torques, the total angular momentum

$$\vec{J} = \vec{J}_1 + \vec{J}_2 + \ldots + \vec{J}_n$$

remains conserved. However $\vec{J}_1, \vec{J}_2, \ldots, \vec{J}_n$ may change individually.

Example : 1. The angular momentum remains conserved in the motion of planets and satellites.

2. The angular momentum remains conserved in the scattering of a proton by a heavy nucleus.

3. An ice skater executing spin.

Q.2. State and prove the theorem of parallel axis.

Ans.

Theorem of Parallel Axis

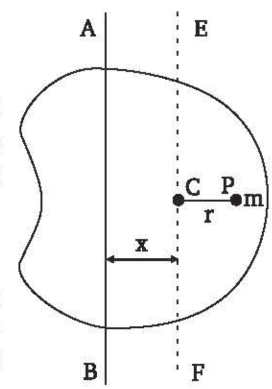


Fig.

This theorem states that the moment of inertia I of a body about any axis is equal to the sum of its moment of inertia about a parallel axis passing its centre of mass I_{cm} and product of its mass and square of the distance x between the two axes *i.e.*,

$$I = I_{cm} + Mx^2$$

Proof : Let C be the centre of mass of a plane lamina as shown in figure. Suppose I be its moment of inertia about an axis AB (in its plane) and I_{cm} is the moment of inertia about an axis EF parallel to AB and passing through C.

Consider a particle P of mass m at distance r from C. Its distance from AB is $(r+x)$ and its moment of inertia about AB is $m(r+x)^2$. The moment of inertia of the whole lamina about AB is,

$$I = \Sigma m(r+x)^2 = \Sigma m(r^2 + x^2 + 2rx) = \Sigma mr^2 + \Sigma mx^2 + \Sigma 2mrx$$

as x is constant, so we can take it out of summation.

So, $$I = \Sigma mr^2 + x^2\Sigma m + 2x\Sigma mr$$

Here, Σmr^2 = moment of inertia of lamina about $EF = I_{cm}$. $\Sigma m = M$, total mass of the lamina

$\Sigma mr = 0$, the sum of moments of all the particles about the axis passing through centre of mass is zero.

Hence, $$I = I_{cm} + Mx^2$$

Q.3. A solid sphere of mass 500 g and diameter 2 cm rolls without slipping with a velocity of 10 cm/s. Calculate its total energy.

Sol. The translational KE of the sphere is

$$K_{trans} = \frac{1}{2}Mv^2$$

and rotational KE is $$K_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{2}{5}MR^2\right)\frac{v^2}{R^2} \qquad \because \omega = \frac{v}{R}$$

$\therefore$ $$\text{Total energy} = K_{trans} + K_{rot} = \frac{1}{2}Mv^2 + \frac{1}{5}Mv^2$$

$$E = \frac{7}{10}Mv^2 \Rightarrow \qquad M = 500\text{ g}, v = 10\text{ cm/s}$$

$\therefore$ $$E = \frac{7}{10} \times 500\text{ g} \times (10\text{ cm/s})^2 = 35000\text{ ergs}$$

$$E = 35000 \times 10^{-7}\text{ joule} = 0.0035\text{ joule}$$

Q.4. State and prove for the theorem of perpendicular axis.

Ans.
Theorem of Perpendicular Axis

This theorem states that the moment of inertia of a plane lamina about a perpendicular axis is equal to the sum of its moments of inertia about any two mutually perpendicular axes in plane of lamina intersecting on the first axis.

Proof : Let OX and OY be two mutually perpendicular axis in the plane of lamina and OZ be the axis perpendicular to the plane of the lamina as shown in the given figure. Obviously the three axis intersecting at O are perpendicular to each other. If I_x and I_y be the moments of inertia of the lamina about OX and OY respectively, then according to the theorem of perpendicular axis,

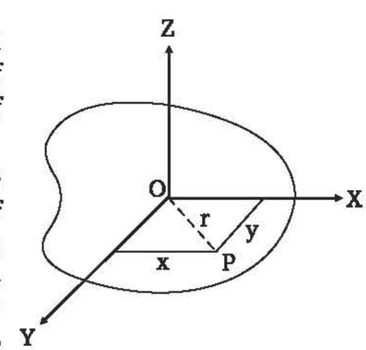


Fig.

$$I_z = I_x + I_y$$

where I_z is the moment of inertia of lamina about the perpendicular axis OZ.

Let us consider a particle P of mass m at a distance r from OZ. If the coordinates of particle are x and y, then its moment of inertia about OX and OY will be my^2 and mx^2 respectively.

The moment of inertia of the particle about $OZ = mr^2$ and of whole lamina $= \Sigma mr^2$. But from the figure $r^2 = x^2 + y^2$,

So, $$I_z = \Sigma m(x^2 + y^2) = \Sigma mx^2 + \Sigma my^2$$

or $$I_z = I_x + I_y$$

Q.5. Find the moment of inertia of a circular disc about (i) a perpendicular axis through its centre, (ii) a diameter.

Ans. **Moment of Inertia of a Circular Disc**

(i) **About an axis through its centre and perpendicular to its plane :** Let M be the mass and R be the radius of the disc (Fig). Mass per unit area of disc $= \dfrac{M}{\pi R^2}$

Consider a concentric elementary ring of radius x and thickness dx.

Area of the ring $= 2\pi x \cdot dx$

$$\text{Mass of the ring} = \frac{M}{\pi R^2} \times 2\pi x \cdot dx = \frac{2M \cdot xdx}{R^2}$$

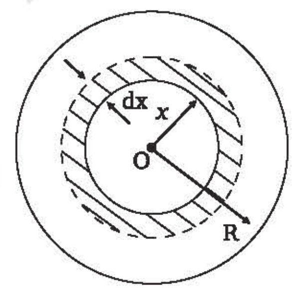


Fig.

Moment of inertia of this ring about an axis through O and perpendicular to its plane

$$= \frac{2Mx\,dx}{R^2} \cdot x^2 = \frac{2Mx^3}{R^2} dx$$

Therefore, moment of inertia of the whole disc about the axis is

$$I = \frac{2M}{R^2} \int_0^R x^3 dx = \frac{2M}{R^2} \left[\frac{x^4}{4}\right]_0^R = \frac{2M}{R^2} \times \frac{R^4}{4}$$

$$I = \frac{1}{2} MR^2$$

(ii) **About a diameter :** Due to symmetry, the M.I. of the disc about any diameter is the same. Using theorem of perpendicular axes, we have

$$I_Z = I_X + I_Y$$

$$\frac{1}{2} MR^2 = I + I \quad \text{or} \quad I = \frac{1}{4} MR^2$$

Q.6. A solid sphere rolls down an inclined plane from rest. Find the velocity after it has suffered a vertical drop of 2 meters ($g = 10$ m/s^2).

Sol. The acceleration down the inclined plane is given by

$$a = \frac{g \sin \theta}{1 + \dfrac{K^2}{R^2}} \quad \text{...(1)}$$

For a solid sphere $I = \frac{2}{5}MR^2 = MK^2$ $\therefore$ $\frac{K^2}{R^2} = \frac{2}{5}$

Putting this equation (1),

$$a = \frac{g\sin\theta}{1+\frac{2}{5}} = \frac{5}{7}g\sin\theta$$

Given, height $h = 2m$. So, $S = \frac{h}{\sin\theta}$

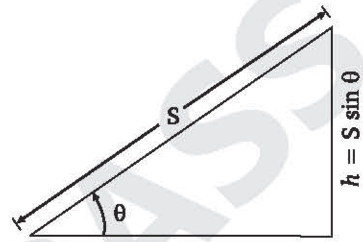


Fig.

The sphere starts from rest, therefore from

$$v^2 = u^2 + 2as, \text{ where } u = 0$$

$$v^2 = 0 + 2\left(\frac{5}{7}g\sin\theta\right) \times \frac{h}{\sin\theta}$$

$$= \frac{10}{7}gh = \frac{10}{7} \times 10 \times 2 = \frac{200}{7}$$

$\therefore$ Velocity attained by the sphere on reaching the bottom is

$$v = \sqrt{\frac{200}{7}} = 5.35 \text{ m/s}$$

Q.7. Find the moment of inertia of a circular ring about (i) a perpendicular axis through its centre, (ii) a diameter.

Ans.

Moment of Inertia of a Thin Circular Ring

(i) About an axis through its centre and perpendicular to its plane : Let M be the mass and R the radius of the circular ring shown in fig. Assume a particle of mass m of the ring. Its moment of inertia about an axis passing through O and perpendicular to its plane is mR^2. Therefore, the moment of inertia of the entire ring about the said axis will be given by

$$I = MR^2 \quad \text{where,} \quad M = \Sigma m.$$

(ii) About any Diameter : Due to symmetry the moment of inertia of a circular ring is the same about any diameter.

Therefore, $I_X = I_Y = I.$

Using theorem of perpendicular axes, we have

$$I_Z = I_X + I_Y$$

i.e., $MR^2 = I + I$

or $I = \frac{1}{2}MR^2$

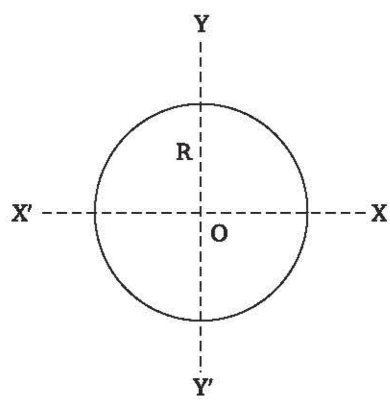


Fig.

Q.8. Write the short note on stress.

Ans.

Stress

When external deforming forces act on a body, internal forces opposing the former are developed at each section of the body. The magnitude of the internal forces per unit area of the section is called "stress". In the equilibrium state of the deformed body, the internal forces are equal and opposite to the external forces. Therefore, stress is measured by the external forces per unit area of their application. Stress is not a vector quantity since, unlike a force, we cannot assign to it a specific direction. **Stress** is one of a class of physical quantities called **tensors.**

The SI unit of stress is the Newton per square meter (N-m^{-2}). This unit is also known as pascal (P_a). Its dimensional formula is $[ML^{-1}T^{-2}]$.

The stress S can be defined as the ratio of the internal restoring force F to the area A;

$$\text{Stress} = F/A$$

The stress is called a tensile stress, meaning that each portion pulls on the other and it is also a normal stress because the distributed force is perpendicular to area.

The stress produced in a body depends upon the manner in which external force is applied. These are of three types :

(i) **Longitudinal Stress :** If deforming force acts along the length of the body, then stress is called longitudinal stress.

(ii) **Normal Stress :** If deforming force acts normal to the surface of a liquid, then stress is called normal stress.

(iii) **Shearing Stress :** If deforming force acts tangentially on the surface of body, then stress is called shearing stress.

Q.9. Write the Hook's law.

Ans. Hooke's Law : Robert Hooke, in 1679, showed experimentally that "provided the strain is small, the stress is proportional to the strain" *i.e.,*

$$\frac{\text{Stress}}{\text{Strain}} = a \text{ constant}$$

This is known as Hooke's law. The constant of proportionality is called "modulus of elasticity". It depends upon the material. It is different for different types of strain in the same material. It's unit is Newton per meter^2 and dimensional formula is $[ML^{-1}T^{-2}]$.

Corresponding to three types of strain, we have three modulli of elasticity namely :

(i) Young's modulus, corresponding to longitudinal strain.

(ii) Bulk modulus, corresponding to volume strain.

(iii) modulus of rigidity, corresponding to the shearing strain.

Q.10. What are the differences between pressure and stress?

Ans. Differences between Pressure and Stress : Pressure is defined as the external force applied on unit area of a surface. When an elastic body is deformed under the action of some applied force, then internal restoring force is developed which opposes change in shape and size of the body.

In equilibrium condition, the internal restoring force is equal to the applied (or external) force. The internal restoring force acting on unit area of the body is called the **stress.** Thus both stress and pressure have the dimensions $[ML^{-1}T^{-2}]$ and their SI unit is Newton/m^2. Mathematically,

$$\text{Pressure (or Stress)} = \frac{F}{A}$$

But for pressure, F is applied force whereas for stress it is the internal restoring force.

Q.11. Define the Young's modulus of elasticity.

Ans. **Young's Modulus**

Within the elastic limits, the ratio of longitudinal stress to the longitudinal strain is called Young's modulus for the material of the body, It is denoted by the letter Y. Let L be the length and A the area of cross section of a wire. Let its length be increased by l when a longitudinal force F is applied along its length.

$$\text{longitudinal stress} = \frac{F}{A}$$

$$\text{longitudinal strain} = \frac{l}{L}$$

Then
$$Y = \frac{F/A}{l/L} = \frac{FL}{Al}$$

Its SI unit is N/m^2 and dimensional formula is $[ML^{-1}T^{-2}]$.

Q.12. Write a short note on modulus of rigidity.

Ans. **Modulus of Rigidity**

Within elastic limits the ratio of tangential stress to the shearing strain is called the modulus of rigidity of the material of the body. It is denoted by the letter η.

Let *abcd* be a section of a block of face area A. Let its lower face be fixed and the upper face is acted upon by a tangential force F. It is sheared and assumes the new shape $ab'c'd$. Let θ be the angle through which its vertical sides have turned.

$$\text{tangential stress} = F/A$$

$$\text{shearing strain} = \theta$$

The modulus of rigidity η of the material of the block is

$$\eta = \frac{F/A}{\theta}$$

The dimensional formula for η is $[ML^{-1}T^{-2}]$.

Q.13. Prove the relation : $Y = 2\eta(1+\sigma)$.

Ans. **Relationship among Y, η and σ**

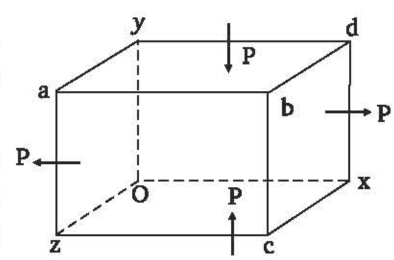


Fig. 1

Consider the compressional stresses on the face *ydba* and *oxcz* parallel to *y*-axis and an equal extensional stress on faces *ayoz* and *bdxc* parallel to *x*-axis (Fig.).

Then extensianal stress P, parallel to x-axis, will produce extension (P/Y) along x-axis, and compression $(\sigma P/Y)$ along each of the y and z-axis. Similarly, compressional stress p parallel to the axis of y will produce compression (P/Y) along the y-axis and extensions $(\sigma P/Y)$ along each of the x and z-axis. The net extensions e_x, e_y and e_z along the three axes of x, y and z are, therefore,

$$e_x = \frac{P}{Y} + \frac{\sigma P}{Y} = \frac{P}{Y}(1+\sigma)$$

$$e_y = -\frac{\sigma P}{Y} - \frac{P}{Y} = -\frac{P}{Y}(1+\sigma)$$

$$e_z = \frac{-\sigma P}{Y} + \frac{\sigma P}{Y} = 0$$

Thus we have equal extension and compression along x and y-axis. But we know that the sum of simultaneous equal compression and extension at right angles to each other are equivalent to shear θ, and hence

$$\theta = \frac{P}{Y}(1+\sigma) + \frac{P}{Y}(1+\sigma) = \frac{2P}{Y}(1+\sigma)$$

or

$$\frac{P}{\theta} = \frac{Y}{2(1+\sigma)}$$

Now, the modulus of rigidity

$$\eta = \frac{\text{shearing stress}}{\text{shearing strain}} = \frac{P}{\theta} = \frac{Y}{2(1+\sigma)}$$

or

$$Y = 2\eta(1+\sigma) \qquad \text{...(2)}$$

This is the required relation between Y, η, σ.

Q.14. What are the applications of bending of beams? Discuss in brief.

Ans.

Applications of Bending of Beams

(a) **In girders of rectangular cross-sections the longer side is used as depth :** The depression $\delta = \left(\frac{MgL^3}{4Ybd^3}\right)$ in the middle of a beam for a given load is directly proportional to the cube of its length (L) and inversely proportional to its breadth (b) and cube of thickness (d). For the depression to be small, the length or span of the girder should be small, while its breadth and depth (or thickness) should be large. Since, the depth varies as cube power, hence an increase in depth reduces the depression much more as compared to the same increase in breadth. Therefore, in the girder of rectangular cross-section, the larger side is used as its depth, so that the girder may not bend appreciably.

(b) **Steel girders and rails are generally *I* shaped :** Girders standing on pillars at their ends support load and suffer bending and its middle part is depressed. The depression produced in the middle of a beam supported at its ends is given by $\delta = \frac{MgL^3}{4Ybd^3}$. In this process the filaments in the upper half are compressed while those in the lower half are extended. These extensions and compressions are greatest near the surface. The stresses produced are also maximum and decreases towards the neutral surface from either side. Therefore, the upper and lower faces of the girder should be much stronger than its inner portions. The inner parts may, therefore, be made of smaller breadth than the upper and lower parts. This is why the steel girders are usually manufactured with their sections in the form of *I*. Thus a large amount of material is saved without sacrificing the strength of the girder.

(c) **Uniform Cantilever is more likely to break at the fixed end :** The external load $W\ (=Mg)$ applied at the free end of a uniform cantilever exerts a bending torque at each section of the cantilever. Its value is different at different points along the length of the cantilever. At a point distant x from the fixed end, the bending torque will be $W\ (L-x)$, where L is the length of the cantilever. Obviously, at the fixed end (*i.e.*, at $x=0$), the bending torque will be maximum (equal to WL). So the uniform cantilever is more likely to break near the fixed end. At the loaded end (*i.e.*, at $x=L$), the bending torque will be minimum and hence the cantilever is least likely to break there.

Q.15. What do you mean by strain energy in a body?

Ans.

Strain Energy in a Body

Suppose a wire of length L and cross sectional area A is stretched by x on applying a force F along its length. Then linear strain $=\dfrac{x}{L}$.

$$\text{tensile stress} = F/A$$

Young's modulus for the material of the wire is

$$Y = \frac{\text{tensile stress}}{\text{linear strain}} = \frac{F/A}{x/L}$$

So the force F required to stretch the wire through x is

$$F = \frac{YA}{L}x$$

The work required to be done to stretch the wire further through a distance dx is

$$dW = F\,dx = \frac{YA}{L}x\,dx$$

Hence the total work done to stretch the wire from the original length L to $L+x$ *i.e.*, from $x=0$ to $x=x$, will be

$$W = \int_0^x \frac{YA}{L}x\,dx = \frac{YA}{L}\left(\frac{x^2}{2}\right)_0^x = \frac{1}{2}\frac{YA}{L}x^2$$

$$= \frac{1}{2}\left(\frac{Yx}{L}\right)\left(\frac{x}{L}\right)AL$$

or $$W = \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume}$$

Hence the work done per unit volume of the wire is

$$= \frac{1}{2} \times \text{stress} \times \text{strain}$$

This work is stored as the elastic (potential) energy or strain energy in the wire.

Q.16. Find the work done in twisting a steel wire of radius 1 mm and length 0.25 m through 45°. η for steel $= 8\times 10\ \text{N/m}^2$.

Sol. Work done in twisting a cylinder (or wire) through ϕ radians is given by

$$W = \frac{1}{2}C\phi^2 = \frac{1}{2}\left(\frac{\pi\eta R^4}{2L}\right)\phi^2$$

Given, $R = 1\text{ mm} = 1\times10^{-3}$ m, $L = 0.25$ m, and

$$\phi = 45° = \frac{45\times\pi}{180}\text{ radians} = \frac{\pi}{4}\text{ radians}$$

$$W = \frac{1}{2}\times\frac{3.14\times8\times10^{10}\times(1\times10^{-3})^4}{2\times0.25}\times\left(\frac{3.14}{4}\right)^2\text{ joule}$$

$$W = 0.15\text{ joule}$$

SECTION-C LONG ANSWER TYPE QUESTIONS

Q.1. Define moment of inertia and radius of gyration. Also show that equation of motion of a rigid body.

Ans.

Moment of Inertia

The moment of inertia (M.I.) of a body or a system of particles about an axis of rotation is given by

$$I = \Sigma mr^2 = m_1r_1^2 + m_rr_2^2 + \ldots$$

where $m_1, m_2, \ldots$ are masses of particles and $r_1, r_2, \ldots$ are their distances from the axis of rotation respectively. Its MKS unit is kg. m^2 and dimensional formula is $[ML^2]$. It is a tensor quantity.

Moment of inertia of a body depends upon :

(a) the mass of the body

(b) the distribution of mass about, and its distance from, the axis of rotation.

Since $\tau = I\alpha$

If $\alpha = 1$, then $\tau = 1$

Thus the M.I. of a rigid body about a given axis may be defined as the torque capable of producing unit angular acceleration about that axis.

For a continuous, homogeneous body

$$I = \int r^2 dm$$

where dm is the elementary mass at a distance 'r' from the axis of rotation.

Radius of Gyration

Let us suppose that mass M of a body is concentrated at a distance 'K' from the axis of rotation, then the M.I. of the body about that axis of rotation may be written as

$$I = MK^2$$

or $$K = \sqrt{(I/M)}$$

Thus the radius of gyration 'K' of a body about an axis may be defined as the distance from the axis of rotation at which if the whole mass of the body were to be concentrated, the M.I. of the body about this axis would remain the same as it would have been with the actual distribution of mass.

Equation of Motion of a Rigid Body

Let us consider a rigid body rotating with an angular velocity $\vec{\omega}$ about a fixed axis OK passing through the point O. If $\vec{R}$ is the position vector of any particle P of mass m relative to O, then its linear velocity is given by

$$\vec{v} = \vec{\omega} \times \vec{R} \qquad ...(1)$$

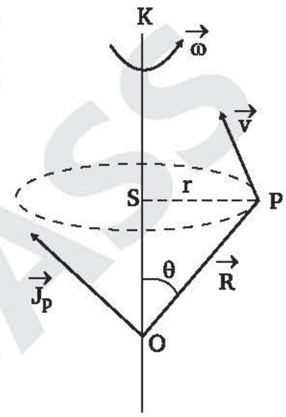


Fig.

Its linear momentum $\vec{p} = m\vec{v}$ and angular moment about O

$$\vec{J_p} = \vec{R} \times \vec{p} = \vec{R} \times m\vec{v} \qquad ...(2)$$

Therefore total angular momentum of the whole body about O is given by

$$\vec{J} = \Sigma \vec{J_p} = \Sigma m(\vec{R} \times \vec{v})$$

$$= \Sigma m \vec{R} \times (\vec{\omega} \times \vec{R}) \qquad ...(3)$$

The direction of $\vec{J}$ will in general not coincide with that of $\vec{\omega}$.

Now, $\vec{J} = \Sigma m \{(\vec{R} \cdot \vec{R})\vec{\omega} - (\vec{R} \cdot \vec{\omega})\vec{R}\}$

or $\vec{J} = \Sigma m \{R^2 \vec{\omega} - (\omega R \cos\theta)\vec{R}\}$

where $\angle POK = \theta$

Hence the magnitude of the component of $\vec{R}$ along $\vec{\omega}$ will be $R\cos\theta$ and then the magnitude of the component (J_0) of $\vec{J}$ along the axis of rotation will be given by

$$J_0 = \Sigma m \{R^2\omega - (\omega R \cos\theta) R\cos\theta\}$$

$$= \Sigma m R^2 \omega (1 - \cos^2\theta) = \Sigma m R^2 \omega \sin^2\theta$$

$$= \omega \Sigma m R_0^2 \qquad ...(4)$$

where $R_0 = R\sin\theta$ is the distance PS and ω is the same for all particles of the body. In equation (4) the $\Sigma m R_0^2$ is called the moment of inertia of the body about the axis of rotation and is represented by I.

Equation (4) can be written as

$$J_0 = I\omega \qquad ...(5)$$

If the axis of rotation and the axis of symmetry of the body is same, then the total angular momentum $\vec{J}$ of the body will be along the axis of rotation *i.e.*,

$$\vec{J} = I\vec{\omega} \qquad \text{or} \qquad J = I\omega$$

The moment of force about O or the torque is given by

$$\vec{\tau} = \frac{d\vec{J}}{dt} = \frac{d}{dt}(I\vec{\omega}) = I\frac{d\vec{\omega}}{dt}$$

or $$\vec{\tau} = I\vec{\alpha} \quad ...(6)$$

where $$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

Equation (6) is the fundamental equation of rigid body.

If the axis of rotation is not the axis of symmetry then $\vec{J}$ will not be along $\vec{\omega}$. Thus for the component of $\vec{J}$ along the axis of rotation, we get

$$\tau = \frac{dJ_0}{dt} = I\frac{d\omega}{dt} = I\alpha \quad ...(7)$$

i.e., the torque about a fixed axis is equal to the product of the moment of inertia (I) and angular acceleration (α) about that axis.

Q.2. Define torque and angular momentum of a particle. Show that the time rate of change of angular momentum of a particle is equal to torque acting on it.

Ans.

Torque Acting on a Particle

We know that a force is required to produce linear acceleration in a body which is in translational motion. In the same way, a torque or moment of force is required to produce angulr acceleration in a body that is in rotational motion. Torque is a vector quantity and its SI unit is newton-meter.

Let a force $\vec{F}$ be applied on a particle P whose position vector relative to origin O of an inertial frame is $\vec{r}$ as shown in fig 1.

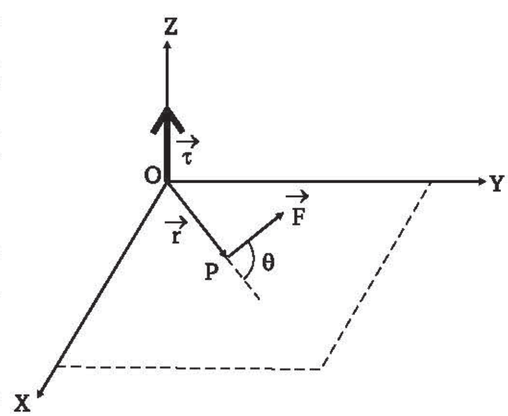


Fig. 1

The torque $\vec{\tau}$ or moment of force $\vec{F}$ on the particle relative to origin O is defined as

$$\vec{\tau} = \vec{r} \times \vec{F} \quad ...(1)$$

Its magnitude is $$\tau = |\vec{\tau}| = rF\sin\theta \quad ...(2)$$

where θ is the angle between $\vec{r}$ and $\vec{F}$. The torque $\vec{\tau}$ is directed perpendicular to the plane of $\vec{r}$ and $\vec{F}$. The direction of $\vec{\tau}$ can easily be determined by using the right hand rule.

Now the magnitude of $\vec{\tau}$ can be expressed in two different ways as

$$\tau = (r\sin\theta)F = r\perp F \quad ...(3)$$

or $\tau = r\,(F \sin\theta) = r \perp F$ ...(4)

The torque will be zero if either

(i) $\theta = 0°$ or $180°$ so that $\sin\theta = 0$, or

(ii) $\vec{r} = 0$ *i.e.*, the particle lies at the origin.

The torque will be maximum when $\theta = 90°$ so that $\vec{r}$ and $\vec{F}$ are perpendicular to each other. In this case

$$\tau_{max} = rF \sin 90° = rF$$

Angular Momentum of a Particle

Let a particle P of mass m whose position vector $\vec{r}$ relative to origin O of an inertial frame, is moving with a velocity $\vec{V}$. The momentum of the particle will be $\vec{p} = m\vec{V}$ (Fig. 2). The angular momentum $\vec{J}$ of the particle relative to origin O is defined as :

$$\vec{J} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{V}$$

$$\vec{J} = m(\vec{r} \times \vec{V}) \quad ...(1)$$

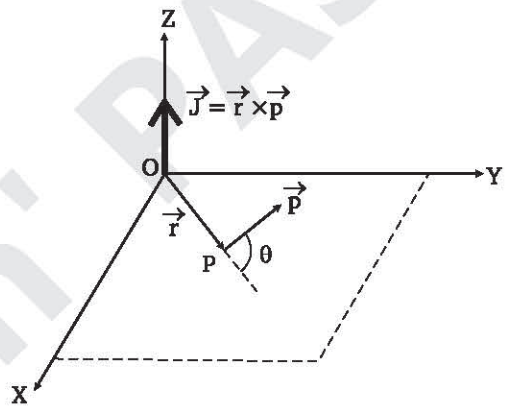


Fig. 2

Thus angular momentum is the moment of linear momentum. It is a vector quantity whose direction is perpendicular to the plane of $\vec{r}$ and $\vec{p}$. Its unit is kg-m^2/s and dimensions are $[ML^2T^{-1}]$ which are also the dimensions of Planck's constant. The direction of $\vec{J}$ can be determined by right hand rule.

The magnitude of angular momentum is

$$J = |\vec{J}| = rp \sin\theta = mvr \sin\theta \quad ...(2)$$

where θ is the angle between $\vec{r}$ and $\vec{p}$.

Both the magnitude and direction of $\vec{J}$ depend upon the choice of origin. The angular momentum of the particle will be (minimum) zero if either :

(i) the line of action of $\vec{p}$ passes through the origin *i.e.*, $\vec{r} = 0$, or

(ii) when $\vec{r}$ and $\vec{p}$ are parallel (collinear) *i.e.*, if $\theta = 0°$ or $180°$.

The angular momentum of the particle relative to O will be maximum when $\theta = 90°$ *i.e.*, when $\vec{r}$ and $\vec{p}$ are perpendicular to each other. Thus

$$\tau_{max} = mvr \sin 90° = mvr \qquad \because \sin 90° = 1$$

Relation between Torque and Angular Momentum

We know that $\vec{J} = \vec{r} \times \vec{p}$

$$\therefore \quad \frac{d\vec{J}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times m\vec{v} + \vec{r} + \frac{d\vec{p}}{dt}$$

$$= 0 + \vec{r} \times \vec{F} \qquad \because \vec{v} \times \vec{v} = 0 \text{ and } \frac{d\vec{p}}{dt} = \vec{F}$$

or $$\frac{d\vec{J}}{dt} = \tau$$

Therefore, $$\vec{\tau} = \frac{d\vec{J}}{dt} = \vec{r} \times \vec{F} \qquad ...(1)$$

Thus the torque acting on a particle is equal to the time rate of change of its angular momentum. Equation (1) plays the same role in rotational motion as Newton's second law $\vec{F} = \frac{d\vec{p}}{dt}$ plays in translational (linear) motion. Equation (1) is valid only when the origin of $\vec{\tau}$ and $\vec{J}$ are same.

Q.3. Derive the expression for the rotational kinetic energy of a body and show that the total kinetic energy of a body of mass M and radius R, rolling without slipping along a plane surface is $\frac{1}{mv}\left(1+\frac{k^2}{R^2}\right)$, where v is the linear velocity of the body and K is its radius of gyration about an axis through its centre of mass.

Ans.

Kinetic Energy of Rotation

When a rigid body performs a linear or translational motion, then its kinetic energy depends on its mass and linear velocity. But, if rigid body is in rotational motion then its kinetic energy depends on its mass and angular velocity as well as on the distribution of its mass relative to its axis of rotation.

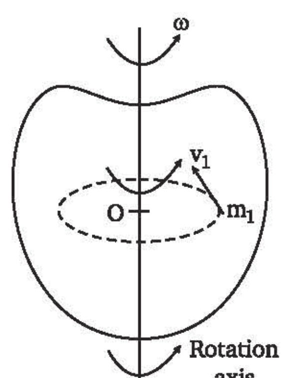


Fig. 1

Consider a rigid body of mass M, moving with an angular velocity ω about an axis passing through O (Fig. 1). If body consist of particles of masses $m_1, m_2, m_3, \ldots$ etc which are at distances $r_1, r_2, r_3, \ldots$ etc. from the axis then angular velocity of all these masses remains the same but linear velocity changes.

Then K.E. of whole body is given by

$$E = \frac{1}{2}m_1 r_1^2 \omega^2 + \frac{1}{2}m_2 r_2^2 \omega^2 + \ldots$$

$$=\frac{1}{2}\omega^2(m_1r_1^2+m_2^2r_2^2+..)=\frac{1}{2}\omega^2\Sigma mr^2$$

$$E=\frac{1}{2}I\omega^2$$

If $\omega=1$, then $I=2E$.

i.e., the moment of inertia of any rigid body is twice the rotational K.E. of the body rotating with unit angular velocity.

Kinetic Energy of a Body Rolling on a Horizontal Plane : Simultaneous Rotation and Translational Motion : Let us consider a body with a circular symmetry of mass M, radius R and with its centre of mass at O, rolling without slipping along a horizontal plane, such that it rotates clockwise and moves along $+x$ direction as shown in fig. 2.

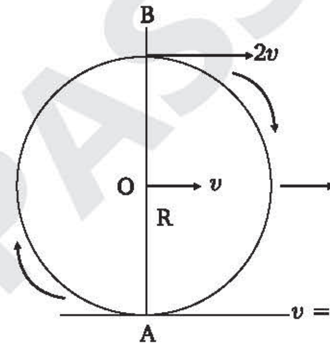


Fig. 2

At any given instant, the point A, is at rest. So an axis through A, perpendicular to the plane of paper, is its instantaneous axis of rotation. The linear velocities of its various particles are perpendicular to the lines joining them with the point of contact A. Their magnitudes are proportional to the lengths of these lines. Thus if linear velocity of the centre of mass O be v, then linear velocity of the particle at point B will be $2v$. This means that the body is rotating about the fixed axis through A with an angular velocity ω, given by (v/R). The motion of body is thus equivalent to one of pure rotation about the axis through A, with an angular velocity ω.

The kinetic energy possessed by the body is its kinetic energy of rotation $\frac{1}{2}I_A\omega^2$, where I_A is its M.I. about the axis passing through the point A.

By the theorem of parallel axis,

$$I_A=I_{CM}+MR^2$$

where I_{CM} is the M.I. of the body about a parallel axis through its centre of mass.

$\therefore$ Kinetic energy of the rolling body,

$$E=\frac{1}{2}I_A\omega^2=\frac{1}{2}(I_{CM}+MR^2)\omega^2$$

$$=\frac{1}{2}I_{CM}\frac{v^2}{R^2}+\frac{1}{2}MR^2.\frac{v^2}{R^2}$$

$$=\frac{1}{2}MK^2\frac{v^2}{R^2}+\frac{1}{2}Mv^2$$

$$E=\frac{1}{2}Mv^2\left(1+\frac{K^2}{R^2}\right)$$

Q.4. A 500 g mass is whirled round in a circle at the end of a string 50 cm long, the other end of which is held in the hand. If the mass makes 8 rev/s, what is its angular momentum? If the number of revolutions is reduced to just

one, after 20 seconds, calculate the mean valueof torque acting on the mass.

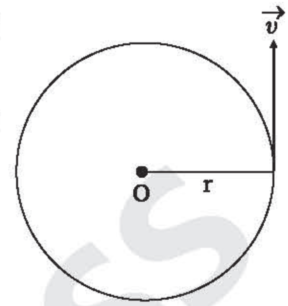


Fig.

Sol. Angular momentum of mass, $\vec{J} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$ which directed perpendicular to the plane of the circle.

So $$|\vec{J}| = J = mvr\sin\theta = mvr\sin 90° = mvr$$

where $\theta = 90°$, is the angle between $\vec{r}$ and $\vec{v}$.

Now, $v = r\omega$ $\quad\therefore\quad J = mr^2\omega$

where angular velocity, $$\omega = 2\pi n = 2\pi \times 8 \text{ rad/s}$$

$\therefore$ Angular momentum, $$J = 500\text{ g} \times (50\text{ cm}^2) \times (2\pi \times 8 \text{ rad/s})$$
$$= 6.28 \times 10^7 \text{ erg-s}$$

Torque, $$\tau = \frac{dJ}{dt} = \frac{d(mr^2\omega)}{dt} = mr^2\frac{d\omega}{dt} = 500 \times (50)^2 \times \frac{(2\pi \times 8 - 2\pi \times 1)}{20}$$
$$= \frac{500 \times 2500 \times 2\pi \times 7}{20}$$
$$\tau = 2.75 \times 10^6 \text{ dyne-cm.}$$

Q.5. Discuss the combined translational and rotational motion of a body over an inclined plane.

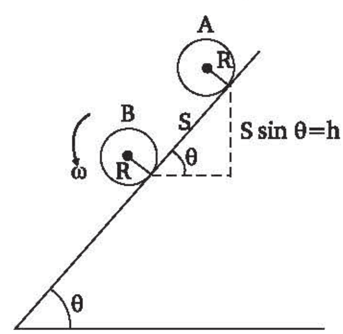


Fig.

Ans. Combined Translational and Rotational Motion on an Inclined Plane : Let a body of mass M and radius R rolls down from rest along an inclined plane without slipping through a distance S as shown in the given figure. The body is under simultaneous translational as well as rotational motions.

$$\text{Gain in KE of translation} = \frac{1}{2}Mv^2,$$

where v is the velocity of body at position B.

$$\text{Gain in KE of rotation} = \frac{1}{2}I\omega^2 = \frac{1}{2}MK^2 \cdot \frac{v^2}{R^2}$$

$\therefore$ $$\text{Total gain in KE} = \frac{1}{2}Mv^2 + \frac{1}{2}Mv^2 \cdot \frac{K^2}{R^2} = \frac{1}{2}Mv^2\left(1 + \frac{K^2}{R^2}\right)$$

Vertical height descended by the body,
$$h = S\sin\theta$$

$\therefore$ $$\text{Loss in PE of the body} = Mgh = MgS\sin\theta$$
$$\text{Gain in KE} = \text{Loss in PE}$$

i.e., $$\frac{1}{2}Mv^2\left(1 + \frac{K^2}{R^2}\right) = mgS\sin\theta$$

or $$v^2 = 2\left[\frac{g\sin\theta}{1+K^2/R^2}\right]S \quad ...(1)$$

Thus, $$v = \sqrt{\frac{2gh}{\left(1+\frac{K^2}{R^2}\right)}}$$

Since the body starts from rest therefore its acceleration '*a*' down the inclined plane is given by

$$v^2 = 2aS \quad ...(2)$$

Comparing (1) and (2), we get

$$a = \frac{g\sin\theta}{1+\frac{K^2}{R^2}} \quad ...(3)$$

It t be the time of descent, then $$S = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2}at^2 \qquad \because u = 0$$

or $$\frac{h}{\sin\theta} = \frac{1}{2}\left[\frac{g\sin\theta}{1+\frac{K^2}{R^2}}\right]t^2 \quad \text{or} \quad t = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1+\frac{K^2}{R^2}\right)} \quad ...(4)$$

which shows that the time of descent is independent of the mass of the body rolling down the plane. From equations (3) and (4) it is clear that the acceleration and the time of descent depend upon radius of gyration K of the body which is different for different shaped objects. Hence, acceleration and time of descent will be different for different shaped objects. Further the time of descent depends upon the inclination θ of the plane.

Q.6. Define various types of strain and also define Poisson's ratio.

Ans.

Strain

When a body suffers a change in its size or shape, under the action of external forces, it is said to be deformed. The term strain refers to the relative change in dimensions or shape of a body that is subjected to stress. Since it is a ratio, the strain has no units and dimensions. Associated with each type of stress is a corresponding type of strain. There are three types of strains :

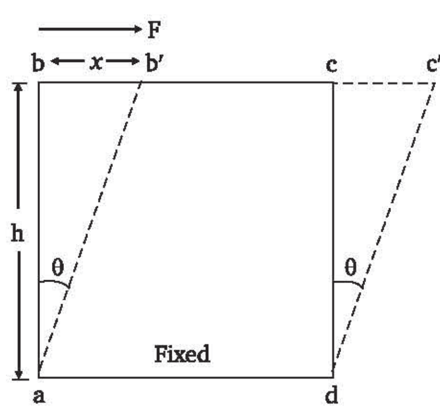


Fig. 1

1. **Longitudinal Strain :** The strain produced in a wire by a force applied along its length is called longitudinal strain.

$$\text{Longitudinal strain} = \frac{\text{Change in length}}{\text{Original length}} = \frac{l}{L}$$

2. **Volume Strain :** The strain produced by a hydrostatic pressure, called a volume strain, is defined as the ratio of the change in volume, ΔV, to the original volume V. It is also a pure number.

$$\text{Volume strain} = \frac{\Delta V}{V}$$

3. **Shearing Strain :** The strain produced by the application of a tangential force is called a shearing strain. It is defined as the ratio of the displacement of any layer to its distance from the fixed surface.

$$\text{Shearing strain} = \frac{x}{h} = \tan\theta = 0 \qquad [\because \theta \text{ is small}]$$

Poisson's Ratio

When a force is applied along the length of a wire then it produces a change not only in its length but also in the dimension perpendicular to the length and vice-versa. The change in length per unit original length is called the linear (or longitudinal) strain.

$$\text{linear strain} = \frac{l}{L}$$

The change produced in the perpendicular (*i.e.*, lateral) dimension per unit lateral dimension is called the lateral strain.

$$\text{lateral strain} = \frac{-d}{D}$$

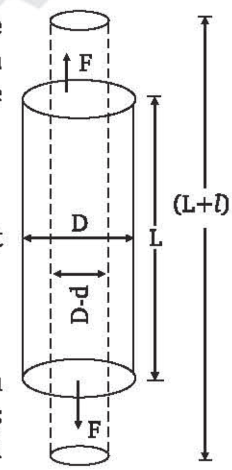


Fig. 2.

Within elastic limits the ratio of lateral strain and longitudinal strain is a constant for the material of a body. This constant is known as the 'Poisson's ratio' and is denoted by the latter σ. It is a pure number and has no units and dimensions since it is the ratio of two strains.

Therefore the poisson's ratio of the material of the wire

$$\sigma = \frac{\text{Lateral strain}}{\text{Linear strain}} = \frac{-d/D}{l/L} = -\frac{lD}{Dl}$$

Negative sign signifies that lateral and linear strains are in opposite sense *i.e.*, if one is positive, the other is negative.

Usually linear and lateral strains per unit stress are denoted by α and β respectively, then

$$\sigma = \frac{\beta}{\alpha}$$

Q.7. Discuss the bending of beams and the terms used in it. Explain bending moment. Deduce an expression for bending moment of a beam in equilibrium position.

Ans.

Bending of Beams

When a beam is bent by an applied torque, tensile force act on some layers of the beam and compressional forces act on some other layers as a result of which filament of the beam nearest the outside curve of the bent beam are extended and the filaments nearest the inside curve get compressed. Internal forces come into play to contact the effect of bending.

The terms used in bending of beams are discussed below :

1. **Beam :** A bar of uniform rectangular or circular cross-section whose length is much greater as compared to its thickness, is called a beam. The shearing stress on its cross-sections become negligibly small. If the beam be fixed only at one end and loaded at the other, it is called a cantilever.
2. **Longitudinal Filament :** A rectangular beam may be supposed to be made up of a large number of thin plane layers placed in contact and parallel to one another. Similarly, a cylindrical beam may be supposed to be made up of thin cylindrical layers placed in contact and coaxial to one another. Further, each layer may be considered as a collection of thin threads (or wires) lying parallel to the length of the beam. These wires are called "longitudinal filaments" of the beams.
3. **Neutral Surface :** When equal and opposite couples are applied at the ends of a beam in a plane parallel to its length, the beam bends into a circular arc. Fig. 1 shows the vertical sections of such a bent beam. In bending, the filaments on the convex side (upper half) of the beam are extended in length and therefore under tension. While those on the concave side (lower half) are compressed and therefore under pressure. These extensions and compressions increase progressively as we move away from the axis on either side. They are therefore maximum in the uppermost and the lowermost layers of the beam respectively. There is, however a plane in the beam in which the filaments remain unchanged in length. This is called the "neutral surface". It passes through the centres of the areas of the cross-sections of the beam.

Fig. 1

4. **Plane of Bending :** The plane in which the beam bends is called the "plane of bending". It is the plane parallel to the long axis of symmetry of the beam. It passes through the axis of symmetry and the centre of curvature of the bent beam.
5. **Neutral Axis :** The line along which the neutral surface of a beam intersects the plane of bending is called the "neutral axis". The extensions and compressions of the filaments are directly proportional to their respective distance from neutral axis.
6. **Bending Moments :** When a beam is bent by an external applied (*i.e.*, bending) couple, an internal restoring couple is developed at each cross-section of the beam due to its elasticity. In the equilibrium state, the restoring couple is equal and opposite to the external couple. The magnitude of the restoring couple is called the "bending moment" for the beam. It is equal to the external couple.

Expression for Bending Moment

When a beam bends by the application of external couple at its ends, then due to elasticity an internal restoring couple produces in each cross-section of the beam, which in equilibrium, is equal to the external couple. The moment of this couple is called as bending moment. In fig. 2, *abcd* is the vertical section of a beam under the action of equal and opposite couples τ, τ at its ends. Let NN' be its neutral axis and beam is divided into two parts by a transverse plane through ef. Due to bending, the filaments of the beam above NN' are elongated, so the forces acting on the left of ef exert a pulling force while

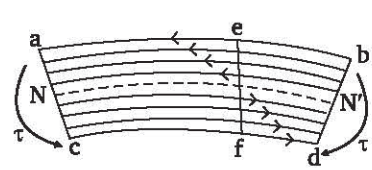


Fig. 2

that on right are pushing forces. The resultant of these two forces form a restoring couple which is in equilibrium with external couple.

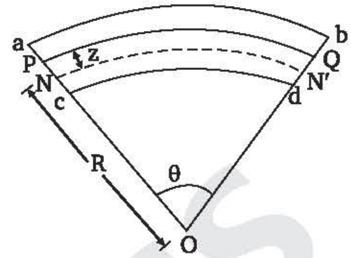


Fig. 3

In fig. 3. a small portion of the beam bounded by two cross sections *ac* and *bd* is shown. Let R be the radius of curvature of the neutral axis NN' and let it subtends an angle θ at its centre of curvature O. Consider a filament PQ at a distance $PN = Z$ from the neutral axis, so that $PO = (R + Z)$.

From fig. 3, we have arc $NN' = R\theta$

and $$PQ = (R + Z)\theta$$

Before bending, the length of the filament PQ was equal to $NN' = R\theta$. Hence the extension in the filament $PQ = (R + Z)\theta - R\theta = Z\theta$.

Hence, longitudinal or extensional strain for PQ is

$$= \frac{\text{increase in length}}{\text{original length}} = \frac{Z\theta}{R\theta} = \frac{Z}{R}$$

Now, $$\text{longitudinal stress} = Y \times \text{longitudinal strain} = \frac{YZ}{R}$$

Let A be the area of cross-section of PQ, then

$$\text{Force acting on this area} = \text{stress} \times \text{area} = \frac{YZ}{R} A$$

The moment of this force about a line through NN' is given by

$$\left(\frac{YZ}{R} A\right) \times Z = \frac{YA}{R} Z^2$$

Now, internal bending moment

$$G = \Sigma \frac{YA}{R} \cdot Z^2 = \frac{Y}{R} \Sigma AZ^2 = \frac{YI}{R}$$

where $I = \Sigma AZ^2$ is called geometrical moment of inertia of the cross-section about an axis through its centroid and perpendicular to the plane of bending.

The product of Young's modulus of the material of the rod and its geometrical moment of inertia *i.e.*, the product YI is called the flexural rigidity.

Q.8. Find an expression for torsional rigidity of a cylinder. Show that a hollow cylinder is stronger as compared to solid cylinder of the same mass, length and material.

Ans.

Torsional Rigidity of a Cylinder

Let us consider a cylindrical rod of length L and radius R whose upper end is kept fixed and a twisting couple is applied at its lower end in a plane perpendicular to its length (Fig. 1a). Due to elasticity, a restoring couple is set up in the rod to oppose the twisting couple.

Let us consider the cylinder to consist of a large number of thin coaxial cylindrical shells and consider one such shell of radius x, and thickness dx. As the rod is twisted, a line AB (parallel to its axis) on the surface of the cylindrical shell takes up the position AB' where $BAB' = \theta$. The angle θ is called the angle of shear. In the equilibrium state, let the angle of twist ($\angle BO'B'$) at the lower end of the rod be ϕ. The angle of twist is zero at the fixed end. Its value increases as we move away from the fixed end. It is maximum at the free end.

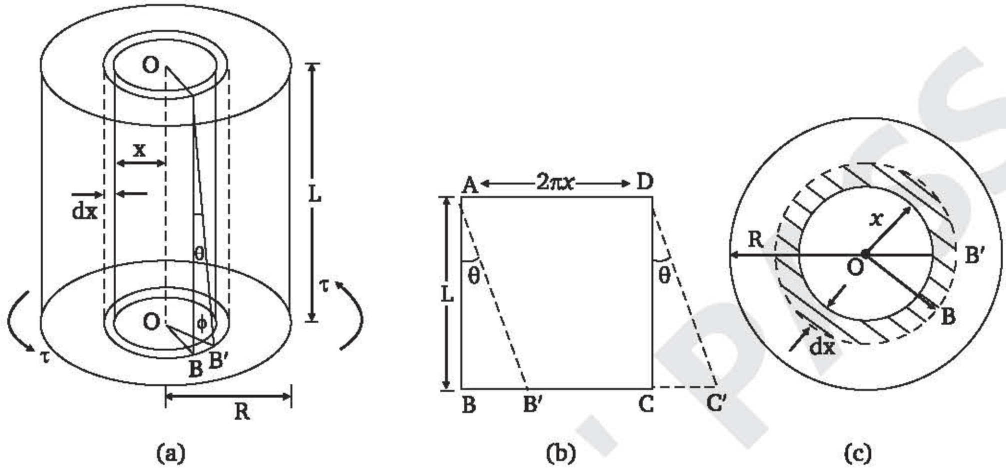


Fig. 1.

Let this cylindrical shell were cut along AB. Before twisting, it will form a rectangular plate $ABCD$, but after twisting it acquires the shape of parallelogram $AB'C'D$ as shown in fig. 1 (b).

Now from $\Delta BO'B'$, arc BB' = angle × radius $= \phi \times x$

Also in $\Delta ABB'$ $\quad BB' = L \cdot \theta$

Hence $\quad L\theta = \phi x$

or
$$\theta = \frac{x\phi}{L} \qquad \text{...(1)}$$

Since the maximum value of x is equal to the radius R of the cylinder, hence the maximum value of shearing strain

$$\theta_{max} = \frac{R\phi}{L}$$

Let F be the tangential force acting over the base of the elementary shell of area $2\pi x\, dx$.

Then,
$$\text{tangential stress} = \frac{F}{\text{Area of the base of the shell}}$$
$$= \frac{F}{2\pi x\, dx}$$

If η is the modulus of rigidity, then

$$\eta = \frac{\text{Tangential stress}}{\text{Shearing strain}}$$

$$\eta = \frac{F/2\pi x\, dx}{\theta} = \frac{F}{2\pi\, x\, dx} \cdot \frac{L}{x\phi}$$

from where $$F = \frac{2\pi\eta\phi}{L} x^2 dx$$

The moment of this force about $$O' = \frac{2\pi\,\eta\phi}{L} x^2 dx \cdot x = \frac{2\pi\eta\phi}{L} x^3 dx$$

This is equal to the couple required to twist the shell of radius x and thickness dx through an angle ϕ. Now, twisting couple on the whole rod

$$\tau = \int_0^R \frac{2\pi\eta\phi}{L} . x^3 dx = \frac{2\pi\eta\phi}{L} \cdot \frac{R^4}{4}$$

or $$\tau = \frac{\pi\eta R^4}{2L}\phi \qquad i.e., \qquad \tau \propto \phi \qquad ...(2)$$

Particularly for $\phi = \frac{\pi}{2}$, we have

$$\tau = \frac{\pi\eta R^4}{2L} \times \frac{\pi}{2} = \frac{\pi^2 \eta R^4}{4L}$$

The couple required to twist the rod through one radian is given by

$$C = \frac{\tau}{\phi} = \frac{\pi\eta R^4}{2L} \qquad ...(3)$$

Twisting couple per unit twist is called torsional rigidity 'C' of the material of the rod. Now, the work done in twisting the wire through an angle $d\phi$ is given by

$$dW = \tau d\phi = C\phi \cdot d\phi$$

Hence, total work done $$W = \int_0^\phi C\phi\, d\phi = \frac{1}{2} C\phi^2 = \frac{\pi\eta R^4 \phi^2}{4L} = \text{strain energy}$$

Hollow Cylinder : A Better Shaft

The couple required to twist a solid cylinder of length L, radius R and modulus of rigidity η, through an angle ϕ is

$$\tau_S = \frac{\pi\eta R^4}{2L}\phi \qquad ...(1)$$

If R_1 and R_2 be the inner and external radii respectively of a hollow cylinder of length L, same mass and material as that of the solid cylinder, then couple required to twist it through the same angle ϕ is

$$\tau_H = \frac{\pi\eta\phi}{2L}(R_2^4 - R_1^4) \qquad ...(2)$$

Now, $$\frac{\tau_H}{\tau_S} = \frac{(R_2^4 - R_1^4)}{R^4} = \frac{(R_2^2 + R_1^2)(R_2^2 - R_1^2)}{R^4}$$

But the cylinders are of the same mass, hence

$$\pi (R_2^2 - R_1^2) L\rho = \pi R^2 L\rho$$

where ρ is the density of the material of the both cylinders.

Now, we have $$R_2^2 - R_1^2 = R^2$$

or $$R_2^2 = R_1^2 + R^2 \quad \text{...(3)}$$

∴ $$\frac{\tau_H}{\tau_S} = \frac{(R_1^2 + R_2^2) R^2}{R^4} = \frac{R_1^2 + R_2^2}{R^2}$$

Adding R_1^2 to both the sides of eqn. (3), we have

$$R_2^2 + R_1^2 = R^2 + 2R_1^2$$

i.e., $$R_2^2 + R_1^2 > R^2$$ because R_1^2 is always positive.

Thus, $$\frac{\tau_H}{\tau_S} > 1$$

i.e., $$\tau_H > \tau_S$$

Therefore, the couple required to twist the hollow cylinder is greater than the couple required to twist a solid one through the same angle. Hence a hollow cylinder is stronger and a better shaft than a solid one of the same mass, material and length.

❑

UNIT-VII

Motion of Planets and Satellites

SECTION-A VERY SHORT ANSWER TYPE QUESTIONS

Q.1. What do you mean by gravitational force?

Ans. Gravitational Force : The force of attraction between all masses in the universe, especially the attraction of the earth's mass for bodies near its surface. It is by far the meakest known force in nature.

Q.2. Define the reduced mass of hydrogen atom.

Ans. Reduced Mass of Hydrogen Atom : In a hydrogen atom both the electron and the proton revolve round their common centre of mass with same angular velocity. This two-particle system can be treated as a one-particle system *i.e.*, the electron with reduced mass revolving around the fixed nucleus (proton). The reduced mass of hydrogen atom is

$$\mu = \frac{m \cdot M}{m + M} = \frac{m}{\left(1 + \frac{m}{M}\right)}$$

where, m = mass of electron

and M = mass of proton $= 1836 \times m$

Since $\frac{m}{M} = \frac{1}{1836}$ $\quad \therefore \; 1 + \frac{m}{M} \approx 1$

or $\mu = m$

Q.3. What is the meant by of central force?

Ans. A central force is a force (possibly negative) that points from the particle directly towards a fixed point in space the centre and whose magnitude only depends on the distance of the object to the centre. Thus

$$F = F(r)\hat{r}$$

F = conservative central force

r = vector magnitude $|r|$ is the distance to the distance to the centre of force

$$\hat{r} = \frac{r}{r}$$

Q.4. What do you understand by reduced mass of positronium atom.

Ans. Reduced Mass of Positronium Atom : This is a temporary hydrogen like combination of an electron and a positron. The positron is the antiparticle of an electron that has the same mass as that of the electron but has a charge of $+e$. The reduced mass of positronium is

$$\mu = \frac{m \cdot m}{m+m} = \frac{m}{2}$$

where m = mass of electron or position.

Thus the reduced mass of positronium is about one-half that of the hydrogen atom.

According to Bohr's theory the frequencies of the spectral lines of a hydrogen-like system of reduced mass μ are given by

$$\nu = \frac{2\pi^2 \mu e^4}{h^3}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) \Rightarrow \nu \propto \mu \text{ or } \lambda = \frac{1}{\mu}$$

where e is the electronic charge, h is the Planck's constant and n_1 and n_2 are integers. Clearly,

(i) The frequencies of the spectral lines of positronium will nearly be half the frequencies of the corresponding lines of hydrogen, and

(ii) The wavelengths of the spectral lines of positronium will nearly be double the wavelengths of the corresponding lines of hydrogen atom.

Q.5. What do you mean by gravitational potential?

Ans. Gravitational Potential : The scalar quantity characteristic of a point in a gravitational field whose gradient equals the intensity of the field and equal to the work required to move a body of unit mass from given point to a point infinitely remote. thus,

$$PE_G = mg\Delta h$$

where, PE_G = potential energy due to gravity

g = Acceleration due to gravity

m = Mass

Δh = Distance above a surface (such as the ground)

Q.6. What is Kepler's law of planetary motion?

Ans. Kepler's law states that the planets move around the sun in elliptical orbits with the sun at one focus.

Q.7. How does a geo-synchronous orbit differ from a geo-synchronous satellite?

Ans. Geo-Synchronous Orbit and Geo-Synchronous Satellites : A geo-synchronous orbit is defined as the orbit in which an object has an orbital period exactly equal to one sidereal day *i.e.*, about 24 hours. A satellite in such an orbit is called a geo-synchronous satellite. Such a satellite returns to the same point above the earth's surface every sidereal day irrespective of other orbital properties. The characteristics of object's path in such an orbit depend upon the orbit's inclination and eccentricity. The orbital plane for a typical geo-synchronous satellite is in general, not the equatorial plane. It can have any inclination. A circular geo-synchronous orbit lies at an altitude of around 36000 km. All geo-synchronous orbits have the same semi-major axis equal to around 42000 km.

Geo-synchronous satellites are mainly communication satellites. The first geo-synchronous satellite was designed by **Harold Rosen.** A special type of geo-synchronous satellites are called geo-stationary satellites.

The key difference between the two types of orbits is that a geo-stationary orbit lies in the equatorial plane whereas a geo-synchronous orbit can have any inclination.

Q.8. Calculate the reduced mass of an α-particle.

Sol. An α-particle is made up of two protons and two neutrons. If m_p and m_n respectively denote the mass of a proton and neutron, then reduced mass of α-particle is

$$\mu = \frac{2m_p \cdot 2m_n}{2m_p + 2m_n} = \frac{2m_p \cdot m_n}{m_p + m_n}$$

$$= \frac{2 \times 1.6725 \times 10^{-27} \times 1.6748 \times 10^{-27}}{[1.6725 + 1.6748] \times 10^{-27}} \text{ kg}$$

$$\mu = 1.6734 \times 10^{-27} \text{ kg}$$

Q.9. Write the relationship between orbit and escape speed.

Ans. Relation between Orbital Speed and Escape Speed : The escape speed for a body projected from the surface of the earth is given by

$$v_e = \sqrt{2gR} \quad ...(1)$$

where g is the acceleration due to gravity and R is the radius of the earth.

The orbital speed of a satellite around earth is

$$v_0 = R\sqrt{\frac{g}{R+h}}$$

where h is the height of the satellite above earth's surface. If $h << R$, then

$$R + h \approx R$$

So,
$$v_0 = R\sqrt{\frac{g}{R}} = \sqrt{gR} \quad ...(2)$$

From equation (1) and (2), we get

$$v_e = \sqrt{2}v_0 = 1.41\, v_0 \quad ...(3)$$

Q.10. What is meant by GPS System?

Ans. The **Global Positioning System (GPS)** is a US owned utility that provides users with positioning, navigation and timing (PNT) services. This system consists of three segments, the space segment the control segment and the user segment.

SECTION-B SHORT ANSWER TYPE QUESTIONS

Q.1. What is a central force? Explain with example.

Ans.

Central Force

If a force acting on a particle is always directed towards or away from a fixed centre and its magnitude depends only upon the distance from the fixed centre, then the force is known as a central force. By the fixed centre, we mean the heavier of the two bodies between which the interaction is being considered. For example, if we consider the motion of the earth around the sun then sun is taken as the fixed centre. Similarly, for the motion of satellites around the earth, the earth is taken as the fixed centre of force. The motion of electrons about the nucleus in an atom is also governed by the central force.

A central force is expressed as :

$$\vec{F}(r) = \hat{r}\, f(r);\ \hat{r} = \frac{\vec{r}}{r}$$

where $f(r)$ is a scalar function of distance r between the two interacting bodies and $\hat{r}$ is the unit vector along the line joining them. The function $f(r) < 0$ means force is attractive, whereas $f(r) > 0$ means force is repulsive.

Gravitational, electrostatic, elastic force etc. are some examples of central forces. Such forces are position-dependent forces. This means that their value depends only upon the instantaneous position of the particle with respect to the fixed centre and on nothing else.

Important characteristics of central forces are as follows :

(i) A central force is always conservative.

(ii) Angular momentum of the body remains conserved in central force motion *i.e.*, angular momentum is a constant of motion.

(iii) The torque acting on a body $\left(\vec{\tau} = \frac{d\vec{J}}{dt}\right)$ is zero in central force motion.

(iv) The central force motion lies in a plane.

Q.2. For a particle moving under central force prove that the angular momentum is conserved.

Ans. Constancy of Angular Momentum under a Central Force : Consider a particle subjected to a central force given by

$$\vec{F} = \hat{r}\, f(r) \qquad \text{...(1)}$$

The torque acting on the particle is given by

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{\tau} \times \hat{r}\, f(r) = \left(\vec{r} \times \frac{\vec{r}}{r}\right) f(r)$$

or $\qquad \vec{\tau} = 0 \qquad \because\ \vec{r} \times \vec{r} = 0$

Thus if $\vec{J}$ be the angular momentum of the particle, then

$$\frac{d\vec{J}}{dt} = \vec{\tau} = 0$$

or $\qquad \vec{J} = \text{constant}$

Thus when a particle moves under the action of a central force, the angular momentum relative to the centre of force is a constant of motion.

The earth's angular momentum relative to the sun is a constant of motion. Similarly, in hydrogen atom, the angular momentum of the electron relative to the nucleus is constant.

Q.3. Define gravitational field and potential.

Ans.

Gravitational Field and Potential

Gravitational Field : The area surrounding a body within which other bodies experience its gravitational force, is called its gravitational field. The gravitational force acting on unit mass placed at a distance r from this body gives the intensity (or strength) E of the gravitational field at that point. Thus,

$$\vec{E} = \frac{\vec{F}}{m} = -\frac{(GMm/r^2)}{m}\hat{r}$$

or

$$\vec{E} = -\frac{GM}{r^2}\hat{r} \qquad ...(1)$$

where M = mass of the body whose gravitational field is considered,

and m = mass (negligibly small) of test body placed at that point

$\hat{r}$ = unit vector along $\vec{r}$.

Gravitational Potential : Gravitational potential V at a point distant r in the gravitational field of a body of mass M is equal to the work done in moving a unit mass from infinity to that point *i.e.,*

$$V = -\int_{\infty}^{r} E\,dr = -\int_{\infty}^{r}\left(-\frac{GM}{r^2}\right)dr = -\frac{GM}{r}$$

This is also the potential energy of a body of unit mass placed at a distance r away from the body of mass M. Thus gravitational potential is equal to the potential energy of unit mass at the same point.

It has been assumed that at infinity both the gravitational force and gravitational potential are zero.

Gravitational potential energy of mass m placed at a distance r from the body of mass M will be

$$U = mV = -\frac{GMm}{r}$$

Q.4. What are Kepler's laws of planetary motion?

Ans.

Kepler's Laws of Planetary Motion

The celestial bodies revolving around the sun in their definite orbits are called planets. Kepler gave the following three laws that govern the motion of the planets around the sun.

1. **The Law of Elliptical Orbits :** All planets move in elliptical orbits having the sun as one focus.
2. **The Law of Areas :** The radius vector of the planet relative to the sun sweeps out equal areas in equal times *i.e.,* the areal velocity of the radius vector is constant.

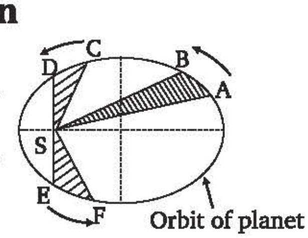


Fig.

3. **The Law of Periods :** The square of the period of revolution (T) of any planet around the sun is proportional to the cube of the semi-major axis ($=a$) of the elliptical orbit *i.e.,*

$$T^2 \propto a^3 \qquad \text{or} \qquad T^2 = Ka^3$$

where K is constant of proportionality.

To understand Kepler's second law consider fig. 1 in which sun *S* is at one focus of the elliptical orbit of the planet. Let the planet takes equal time in moving from *A* to *B* or from *C* to *D* or from *E* to *F*. Then Kepler's second law means that the three areas *SAB* or *SCD* or *SEF* are equal. Thus the areal velocity (*i.e.,* area swept by radius vector per unit time) of the radius vector joining the planet to the sun is constant.

Q.5. Write short note on orbital speed of the satellite.

Ans. Orbital Speed of the Satellite

Let a satellite of mass m is revolving round the earth in a circular orbit of radius *r* with a speed v as shown in fig. The gravitational attraction of the earth provides the necessary centripetal force for the satellite for this motion. Since mass of the earth, $M >> m$, therefore earth is taken to the stationary.

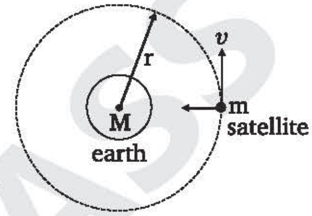


Fig.

$$\therefore \quad \frac{Mv^2}{r} = \frac{GMm}{r^2} \qquad ...(1)$$

or
$$v = \sqrt{\frac{GM}{(R+h)}}, \qquad r = R + h \qquad ...(2)$$

If g be acceleration due to gravity, then $GM = gR^2$, therefore

$$v = R\sqrt{\frac{g}{(R+h)}} \qquad ...(3)$$

If $h << R$, then (3) becomes $v \approx \sqrt{gR}$.

Equation (1), (2) and (3) give the expressions for orbital speed of a satellite.

From equation (1), it is clear that

$$v \propto \frac{1}{\sqrt{r}}$$

Therefore the speed of the satellite increases when satellites jumps to the orbit of smaller radius and its speed decreases when it jumps to the orbit of greater radius.

Q.6. Calculate the escape velocity from the moon. Given that : mass of moon $= 7.4 \times 10^{22}$ kg, radius of the moon $= 1740$ km.

Sol. The escape velocity is given by

$$v = \sqrt{\frac{2GM}{R}}$$

Mass of moon, $M = 7.4 \times 10^{22}$ kg

Radius of the moon, $R = 1740 \text{ km} = 1.74 \times 10^6$ m

$$\therefore \quad v = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 7.4 \times 10^{22}}{1.74 \times 10^6}} = 2400 \text{ m/s}$$

or
$$v = 2.4 \text{ km/sec}$$

This escape velocity is so small that most of the gas molecules have their rms speeds greater than this and hence escaped from moon's gravitational field. This is why there is no atmosphere around the moon.

Q.7. What do you understand by escape velocity?

Ans.

Escape Velocity

The minimum initial velocity with which a particle must be projected upward so as to escape from the earth's gravitational field forever, is called the escape velocity.

The gravitational potential energy of a particle of mass m on the surface of the earth is given by

$$U(R) = -\frac{GMm}{R}, \quad r = R$$

where M and R are the mass and radius of the earth respectively.

The amount of work required against the gravitational force to move the particle from the surface of the earth to infinity would be $\frac{GMm}{R}$. Thus if the kinetic energy of the particle being projected upward is greater than this amount, then it will never return to the earth (provided the resistance of the earth's atmosphere is neglected). Thus the minimum velocity v_e (and minimum $KE = \frac{1}{2}mv_e^2$) above which escape is possible, is given by

$$\frac{1}{2}mv_e^2 = \frac{GMm}{R}$$

or
$$v_e = \sqrt{\frac{2GM}{R}} \quad ...(1)$$

Since, $GM = gR^2$, therefore
$$v_e = \sqrt{2gR} \quad ...(2)$$

It is independent of the particle being projected.

For the earth : $g = 9.8 \text{ m/s}^2$ and $R = 6.4 \times 10^6$ m

$\therefore$
$$v_e = \sqrt{2 \times 9.8 \times 6.4 \times 10^6} \text{ m/s} = 11.2 \times 10^3 \text{ m/s}$$

or
$$v_e = 11.2 \text{ km/sec.}$$

The escape velocity will be different for different planets. Since the root mean square (*rms*) speed of hydrogen molecules in the earth's atmosphere is greater than the escape velocity, therefore the hydrogen gas has escaped from the earth's atmosphere. The escape speed for the sun is much greater than the *rms* speed of hydrogen molecules, so none of hydrogen can escape from the surface of the sun. The escape velocity for the moon is so small that almost every gas can escape from it. Thus there is no atmosphere at the moon.

Q.8. What is geo-stationary satellite?

Ans.

Geo-stationary Satellites

Satellites used for communication purposes such as INSAT, transmit signals over long distances. They receive these signals from one point on earth's surface and then reflect them to another point. To send signals at a particular place (target), the satellite must remain stationary at a point above the earth.

The earth rotates about its own axis from west to east with a time period of 24 hours. Thus a satellite would appear stationary over a point on earth's equator if (i) its orbit is equatorial and circular, and (ii) its period of revolution is 24 hours. Then the revolution of the satellite would be synchronous with earth's rotation. Such a satellite is known as "Geo-stationary" satellite and the orbit in which it revolves around the earth is called geo-stationary orbit.

A geo-stationary orbit is a special case of geo-synchronous orbit. Thus a geo-stationary orbit is a circular geo-synchronous orbit in earth's equatorial plane.

The period of revolution for a satellite about earth is

$$T = \frac{2\pi (R+h)^{3/2}}{R\sqrt{g}} \quad ...(1)$$

Clearly T increases with increase in the height h of the satellite above earth.

For Geo-stationary satellite :

$$T = 24 \text{ hours} = 24 \times 60 \times 60 \text{ sec} = 86400 \text{ sec}$$

$$R = \text{Radius of earth} = 6.4 \times 10^6 \text{ m}$$

$$g = 9.8 \text{ m/s}^2$$

Substituting these values in equation (1), we get

$$h \approx 36{,}000 \text{ km}$$

Thus for a satellite to be geo-stationary, its orbit must be equatorial and should lie at a height of approximately 36,000 km above the earth's surface.

The advantage of geo-stationary satellites is that receiving and transmitting antennas on the earth do not need to track them. These satellites provide high temporal resolution data. Their disadvantages are :

1. High altitude orbit which results in the delay of radio signals, and
2. Incomplete geographical coverage.

Some of their applications are :

1. In communications
2. Television broadcasting
3. Weather forecasting
4. In defense and intelligence.

Q.9. A satellite is in a circular orbit around the earth. Find the minimum additional energy needed by the satellite to escape from the earth's gravitational field.

Sol. The kinetic energy of the satellite of mass m in the circular orbit is given by

$$K = \frac{1}{2} m v_0^2 \quad ...(1)$$

where v_0 is the orbital speed of the satellite.

The energy needed by the satellite to escape from earth's gravitational field is $\frac{1}{2} m v_e^2$, where v_e is the escape velocity. Therefore the additional energy required is

$$= \frac{1}{2} m v_e^2 - \frac{1}{2} m v_0^2 = \frac{1}{2} m (\sqrt{2} v_0)^2 - \frac{1}{2} m v_0^2 \qquad \because v_e = \sqrt{2} v_0$$

$$= \frac{1}{2} m (2v_0^2 - v_0^2) = \frac{1}{2} m v_0^2 = K$$

Thus the kinetic energy of the satellite must be suddenly doubled for its escape.

Q.10. Write short note on global positioning system (GPS).

Ans. **Global Positioning System (GPS)**

Global positioning system (GPS) is a satellite-based radio navigation system used for locating objects in real-time. It is a system consisting of 24 satellites constantly orbiting around the earth in six different approximately circular orbits, each consisting of four satellites. GPS is owned and managed by United States (US) government but provides access to anyone free-of-cost. It provides geo-location and information about time and velocity to a GPS receiver placed near or on the earth lying in the view of four or more satellites at a time. This information about a GPS receiver is based on data received by it from multiple GPS satellites.

If the GPS receiver is in the field of view of three GPS satellites, then the latitude and longitude information can be obtained about it. But if the GPS receiver is in the view field of four satellities, then the altitude can also be determined in addition to latitude and longitude. The GPS does not need any communication device or internet connectivity, however the additional availability of these components enhance usefulness of GPS. It became fully operational in 1993. GPS finds applications in diverse fields such as farming, construction, mining, surveying, logistic supply chain etc.

GPS satellites revolve in the orbits lying at an altitude of 20,000 km above the earth and have orbital speeds about 14000 km/h. Its advantages include :

- freely available everywhere on the globe and easy to use.
- provides location-based information which is quite useful in tracking vehicles, locating desired destinations etc.

Some disadvantages associated with GPS are :

- GPS services can't be availed indoors or under water or in underground stores etc.
- Effects of multipath, electromagnetic interference, atmosphere etc. on GPS signal lead to an error of about 5 to 10 m in location determination.

SECTION-C (LONG ANSWER TYPE) QUESTIONS

Q.1. Show that a two particle problem under central force can always be reduced to equivalent one particle problem. What is reduced mass?

Ans. **Two-Particle Central Force Problem**

Two-particle central force problem can always be reduced to an equivalent one-particle problem. To prove this, let us consider two particles of masses m_1 and m_2 whose position vectors with respect to an arbitrary origin O in an inertial reference frame are $\vec{r}_1$ and $\vec{r}_2$ respectively (Fig.).

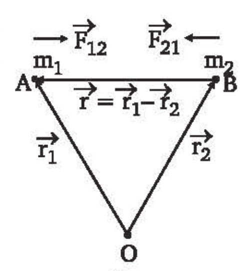


Fig.

$$\vec{r}=\vec{r}_1-\vec{r}_2 \quad ...(1)$$

The particles exert gravitational forces of attraction on each other which act along the vector $\vec{r}$ and thus are central forces. Let $\vec{F}_{12}$ and $\vec{F}_{21}$ be the forces acting on particle m_1 (due to m_2) and m_2 (due to m_1) respectively. Then the equations of motion (Newton's second law) for masses m_1 and m_2 with respect to origin O are respectively.

$$m_1 \frac{d^2 \vec{r_1}}{dt^2} = \vec{F_{12}} \quad \text{and} \quad m_2 \frac{d^2 \vec{r_2}}{dt^2} = \vec{F_{21}}$$

But from Newton's third law

$$\vec{F_{12}} = -\vec{F_{21}} = \vec{F} \text{ (say)}$$

$$\therefore \qquad \frac{d^2 \vec{r_1}}{dt^2} = \frac{\vec{F}}{m_1} \quad \text{and} \quad \frac{d^2 \vec{r_2}}{dt^2} = -\frac{\vec{F}}{m_2}$$

Substracting these equations, we get

$$\frac{d^2 \vec{r_1}}{dt^2} - \frac{d^2 \vec{r_2}}{dt^2} = \frac{\vec{F}}{m_1} + \frac{\vec{F}}{m_2}$$

or
$$\frac{d^2 (\vec{r_1} - \vec{r_2})}{dt^2} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}$$

or
$$\frac{d^2 \vec{r}}{dt^2} = \frac{\vec{F}}{\mu}$$

or
$$\mu \frac{d^2 \vec{r}}{dt^2} = \vec{F} \qquad \text{...(2)}$$

where
$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

or
$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad \text{...(3)}$$

is called the reduced mass of the two particles having masses m_1 and m_2. Equation (2) represents the equation of motion of a single particle of mass μ placed at a vector distance $\vec{r}$ from the fixed centre (B) which exerts on it a central force $\vec{F}$. Thus the original two-particle problem involving two vectors $\vec{r_1}$ and $\vec{r_2}$ has been reduced to a one-particle problem involving a single vector $\vec{r}$.

For the motion of moon about the earth, we take the earth as the fixed centre and the motion of moon can be described by considering the motion of a body having mass equal to the reduced mass for this system. Similarly, the motion of an electron in an atom about the nucleus is equivalent to the motion of a body having mass equal to the reduced mass of the electron-proton system, considering the proton as fixed centre. With regard to the reduced mass, we note that :

(i) It is lesser than either mass *i.e.,* $\mu < m_1$ and $\mu < m_2$.

Since $$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{or} \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

$\therefore$ $$\frac{1}{\mu} > \frac{1}{m_1} \quad \text{and} \quad \frac{1}{\mu} > \frac{1}{m_2}$$

i.e., $$\mu < m_1 \quad \text{and} \quad \mu < m_2.$$

(ii) For a given total mass, the reduced mass of a two-particle system is maximum when the two masses are equal.

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{\sqrt{(m_1 - m_2)^2 + 4m_1 m_2}}$$

For μ to be maximum, the denominator should be minimum for which we must have

$$m_1 - m_2 = 0$$

or $$m_1 = m_2 = m \text{ (say)}$$

Then $$(\mu)_{max} = \frac{m \cdot m}{m + m} = \frac{m}{2}$$

(iii) If there is a great difference between the masses of two particles, then the reduced mass is nearly equal to the smaller mass. To show this, let $m_1 < m_2$

then $$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1}{\left(1 + \frac{m_1}{m_2}\right)} \quad \text{or} \quad \mu \approx m_1$$

because $$1 + \frac{m_1}{m_2} \approx 1 \quad \because \quad m_1 << m_2$$

Reduced Mass : When two bodies in relative motion are acted upon by a central force involving Newton's law then the system can be replaced by a single mass called the **reduced mass. Example :** A light spring of force constant K is held between two blocks of masses m and $2m$. It is represent by μ.

$$\mu = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}}$$

where, μ = reduced mass, m_1 = mass of the 1st body, m_2 = mass of the 2nd body.

Q.2. State and explain the Newton's law of universal gravitation. Discuss its limitations.

Ans.

Newton's Law of Universal Gravitation

Newton in 1686 gave a universal law of force acting between any two material particles. The law may be stated as, "Every two material particles of universe attract each other with a force that is proportional to the product of their masses and inversely proportional to the square of distance between them."

If the particles of masses m_1 and m_2 are at a distance r from each other, then the magnitude of force of attraction between them is

$$F \propto \frac{m_1 \cdot m_2}{r^2}$$

or
$$F = \frac{Gm_1 m_2}{r^2} \qquad \text{...(1)}$$

where the constant of proportionality G, is called the **universal gravitational constant.**
Now from equation (1)

$$G = \frac{Fr^2}{m_1 \cdot m_2}$$

thus if $m_1 = m_2 = 1$ kg, $r = 1$ m, then $G = F$.
Thus the gravitational constant G is numerically equal to the force of attraction between two particles each of mass 1 kg and at a distance 1 m from each other. The dimensions of G may be obtained as follows :

$$[G] = \frac{[\mathrm{MLT}^{-2}]\,[\mathrm{L}^2]}{[\mathrm{M}]\,[\mathrm{M}]} = [\mathrm{M}^{-1}\mathrm{L}^3\mathrm{T}^{-2}]$$

Its value in SI system is 6.67×10^{-11} $\mathrm{Nm}^2/\mathrm{kg}^2$. We may note that :

(i) The gravitational constant G is same everywhere and its value is independent of temperature, pressure or nature of bodies.

(ii) In our daily life we do not experience observable gravitational force due to extremely small value of G. However, due to their greater masses, celestial bodies (*e.g.,* earth, sun, moon etc.) experience appreciable gravitational force between them. The gravitational force supplies the necessary centripetal force for planets and satellites to move in circular orbits.

(iii) The Newton's law of gravitation holds only for point masses. However it may be applied to actual bodies whose dimensions are negligible as compared to their separation. At very short distances between particles (< 10 Å), cohesive and adhesive forces come into play and restrict the applicability of this law.

With regard to gravitational forces between two particles we note that :

(i) these forces are always attractive (F is negative)

(ii) are central forces *i.e.,* act along the line joining them

(iii) form action-reaction pairs, and

(iv) are independent of the medium between the particles.

Vector form of Newton's Law

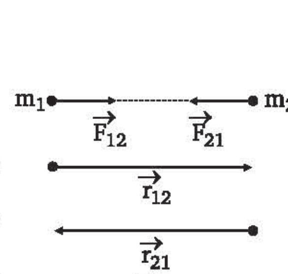


Fig.

We consider two particles of masses m_1 and m_2 as shown in fig. 1. The displacement vector $\vec{r}_{12}$ point from the particle of mass m_1 to the particle of mass m_2 and $\vec{r}_{21}$ that in the reverse direction. The gravitational force on particle m_2 due to m_1 is

$$\vec{F}_{21} = -G\frac{m_1 m_2}{r_{12}^3}\vec{r}_{12} \qquad \text{...(1)}$$

Here r_{12} is the magnitude of $\vec{r_{12}}$ and the negative sign shows that $\vec{F_{21}}$ is in the opposite direction of $\vec{r_{12}}$. If $\hat{r}$ be the unit vector along $\vec{r_{12}}$ *i.e.*, $\hat{r} = \frac{\vec{r_{12}}}{\vec{r_{12}}}$, then

$$\vec{F_{21}} = -G\frac{m_1 m_2}{r_{12}^2}\left(\frac{\vec{r_{12}}}{r_{12}}\right) = -G\frac{m_1 m_2}{r_{12}}\hat{r} \qquad ...(2)$$

The force exerted on m_1 due to m_2 is, therefore,

$$\vec{F_{12}} = -G\frac{m_1 m_2}{r_{12}^3}\vec{r_{21}} = -G\frac{m_1 m_2}{r_{21}^2}\left(\frac{\vec{r_{21}}}{r_{21}}\right)$$

or

$$\vec{F_{12}} = -G\frac{m_1 m_2}{r_{12}^2}(-\hat{r}) \qquad ...(3)$$

because $\vec{r_{21}} = -\vec{r_{12}}$ but $|\vec{r_{21}}| = |\vec{r_{12}}|$.

From (2) and (3), we observe that

$$\vec{F_{12}} = -\vec{F_{21}}$$

i.e., both particles experience equal but opposite forces.

Limitations of Newtons Law

Newton's description of gravity is sufficiently accurate for many practical purposes and is therefore widely used. Deviations from it are small when the dimensionless quantities ϕ/c^2 and $(v/c)^2$ are both much less than one, where ϕ is the gravitational potential, v is the velocity of the objects being studied, and c is the speed of light in vacuum. For example, Newtonian gravity provides an accurate description of the Earth/Sun system, since

$$\frac{\phi}{c^2} = \frac{GM_{\text{sun}}}{r_{\text{orbit}}c^2} \sim 10^{-8}, \quad \left(\frac{v_{\text{Earth}}}{c}\right)^2 = \left(\frac{2\pi r_{\text{orbit}}}{(1\text{ yr})c}\right)^2 \sim 10^{-8}$$

where r_{orbit} is the radius of the Earth's orbit around the Sun.

In situations where either dimensionless parameter is large, then general relativity must be used to describe the system. General relativity reduces to Newtonian gravity in the limit of small potential and low velocities, so Newton's law of gravitation is often said to be the low-gravity limit of general relativity.

Q.3. Obtain expressions for gravitational potential due to a spherical shell at (i) outside the sphere, (ii) on the surface, and (iii) inside the shell. Also find gravitational field at these points.

Ans. Gravitational Potential and Field due to a Spherical Shell

These can be discussed as follows :

1. Gravitational Potential

We will find expressions for gravitational potential at the surface, inside and outside points of the spherical shell.

(i) **Outside the shell :** We have to find gravitational potential at point M which is at a distance r away from the centre of the shell whose mass per unit surface area is σ and radius is R.

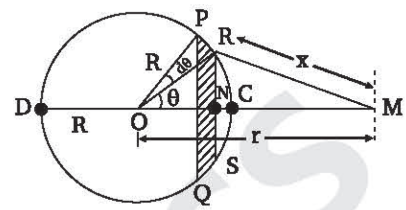


Fig. 1

Consider a ring shaped element $PQRS$ of the shell. Radius of the elementary ring,

$$RN = RO \sin\theta = R\sin\theta$$

$$\text{Its circumference} = 2\pi\,(R\sin\theta)$$

$$\text{width of the ring, } PR = R d\theta$$

$$\text{Mass of the ring} = (\text{area}) \times \sigma = (2\pi R^2 \sin\theta\, d\theta)\,\sigma$$

Every point of this ring is at a distance x from observation point M. Thus potential at M due to this elementary ring $PQRS$, is

$$dV = -\frac{G\,(\text{mass of ring})}{x} = -\frac{G\,(2\pi R^2 \sigma \sin\theta\, d\theta)}{x} \qquad \text{...(1)}$$

In the ΔORM, $\quad x^2 = R^2 + r^2 - 2r\,R\cos\theta$

$\therefore \quad 2x\,dx = -2r\,R\,(-\sin\theta\, d\theta) \qquad r$ & R are constants

or $\quad x = \dfrac{r\,R\sin\theta\, d\theta}{dx}$

Putting this value of x in eqn. (1), we get

$$dV = -\frac{2\pi R^2 G\,\sigma \sin\theta\, d\theta}{r\,R\sin\theta\, d\theta}\,dx = -\frac{2\pi\,R\sigma G}{r}\,dx$$

Min. value of $x = MC = r - R$; Maximum value of $x = MD = r + R$

$\therefore$ Gravitational potential at M due to the whole shell

$$V = \int_{r-R}^{r+R} dV = -\frac{2\pi RG\sigma}{r}\int_{r-R}^{r+R} dx = -\frac{2\pi RG\sigma}{r}\,[x]_{r-R}^{r+R}$$

$$V = -\frac{(4\pi R^2 \sigma)\,G}{r} = -\frac{GM}{r} \qquad \text{...(2)}$$

where $M = (4\pi R^2)\sigma$, is the mass of the spherical shell. Thus for outside points, the shell behaves as if its total mass is concentrated at its centre O.

(ii) **On the surface of the shell :** Let the point M be situated at the surface of the shell (*e.g.*, at C). Then x varies from $MC = 0$ to $MD = 2R$. Hence,

$$V = \int_0^{2R} dV = -\frac{2\pi RG\sigma}{r}\int_0^{2R} dx = -\frac{2\pi RG\sigma}{r}\,[x]_0^{2R}$$

$$V = -\frac{(4\pi R^2 \sigma)\,G}{r} = -\frac{GM}{r} = -\frac{GM}{R} \qquad (\because\ r = R) \quad \text{...(3)}$$

Again the shell behaves as if its total mass is concentrated at the centre *O*.

(iii) **Inside the shell :** Consider now the point *M* lies inside the spherical shell (Fig. 2). Then

$$dV = -\frac{2\pi R\sigma G}{r}\,dx$$

$$V = \int_{R-r}^{R+r} dV = -\frac{2\pi R\sigma G}{r}\int_{R-r}^{R+r} dx = -4\pi R\sigma G = -\frac{(4\pi R^2\sigma)G}{R}$$

$$V = -\frac{GM}{R} \qquad \text{...(4)}$$

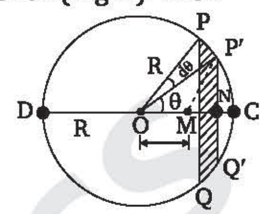


Fig. 2

Thus potential due to the shell at an inside point is the same everywhere and is equal to the value at the surface.

2. Gravitational Field

(i) **At an outside point :** For $r > R$, $V = -\frac{GM}{r}$

$\therefore$ Intensity of gravitational field, $E = \frac{-dV}{dr} = -\frac{d}{dr}\left(-\frac{GM}{r}\right)$

or

$$E = -\frac{GM}{r^2} \qquad E \propto \frac{1}{r^2} \qquad \text{...(5)}$$

The shell behaves as if its total mass is concentrated at its centre.

(ii) **On the surface of shell :** For $r = R$, $V = -\frac{GM}{R} = -\left(\frac{GM}{r}\right)_{r=R}$

$\therefore$

$$E = -\frac{dV}{dr} = -\frac{d}{dr}\left[-\left(\frac{GM}{r}\right)_{r=R}\right] = -\frac{GM}{R^2} \qquad \text{...(6)}$$

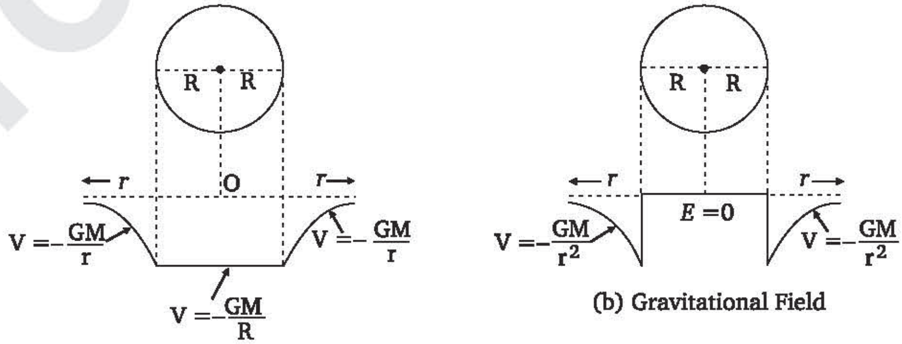
Which is constant everywhere on the surface of the shell.

(iii) **Inside the shell :** For $r < R$, we have seen that gravitational potential is constant every where inside the shell *i.e.*, its value is independent of r. Hence,

$$E = -\frac{dV}{dr} = 0$$

Thus gravitational field strength at all points inside the shell is zero.

A sharp discontinuity exists in the gravitational field intensity at the surface of the shell. On the inner surface its value is zero whereas on the outer surface its value is $-GM/R^2$.



(a) Gravitational Potential

(b) Gravitational Field

Fig. 3 : Gravitational Potential and Field due to a spherical shell

Q.4. State and explain the deduction of Kepler's laws from Newton's laws of gravitation.

Ans.

Deduction of Kepler's Laws from Newton's Law or Gravitation

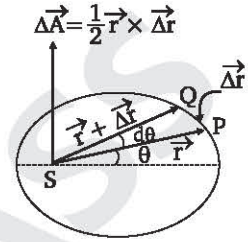


Fig. 1

We consider a planet of mass *m* moving in the gravitational field of the sun. The gravitational force (attractive) between the planet and the sun is given by Newton's law as :

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} \qquad ...(1)$$

where *M* is the mass of the sun and *r* is the distance of the planet from the sun.

Deduction of Second law

The gravitational force is a central force $\left(F \propto \frac{1}{r^2}\right)$. Hence the angular momentum $\vec{J}$ of the planet relative to the sun is conserved in magnitude and direction. Consequently, the motion of the planet must take place in a fixed plane and the areal velocity of its radius vector must be constant as seen below.

Let the planet moves from *P* to *Q* in time interval ΔT (Fig. 1). The vector area $\Delta\vec{A}$ swept by the radius vector in time interval Δt is given by

$$\Delta\vec{A} = \text{area } SPQ$$

$$= \frac{1}{2}\vec{r} \times \Delta\vec{r} \qquad (\because \text{ Arc } PQ = \text{straight line } PQ = \Delta\vec{r})$$

Then

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2}\vec{r} \times \frac{\Delta\vec{r}}{\Delta t}$$

or

$$\lim_{\Delta t \to 0}\frac{\Delta\vec{A}}{\Delta t} = \lim_{\Delta t \to 0}\frac{1}{2}\vec{r} \times \frac{\Delta\vec{r}}{\Delta t} = \frac{1}{2}\vec{r} \times \left(\lim_{\Delta t \to 0}\frac{\Delta\vec{r}}{\Delta t}\right)$$

or

$$\frac{d\vec{A}}{dt} = \frac{1}{2}\vec{r} \times \frac{d\vec{r}}{dt} = \frac{1}{2}\vec{r} \times \vec{V} \qquad \left(\because \frac{d\vec{r}}{dt} = \vec{V}\right)$$

$$= \frac{\vec{r} \times m\vec{V}}{2m} = \frac{\vec{r} \times \vec{p}}{2m}$$

or

$$\frac{d\vec{A}}{dt} = \frac{\vec{J}}{2m} = \text{constant} \qquad ...(2), \quad \text{as } \vec{J} \text{ is constant.}$$

This proves that the areal velocity of the radius vector is constant in planetary motion. This is Kepler's second law.

Deduction of First Law

The magnitude of areal velocity is

$$\frac{dA}{dt} = \frac{J}{2m} = \frac{h}{2} \text{ (say)} \qquad ...(3)$$

where h is a constant.

Also
$$J = I\omega = mr^2 \frac{d\theta}{dt} \qquad ...(4)$$

∴
$$\frac{h}{2} = \frac{1}{2m}\left(mr^2 \frac{d\theta}{dt}\right)$$

or
$$h = r^2 \frac{d\theta}{dt} \qquad ...(5)$$

The radial force on the planet = mass × radial acceleration.

$$F = m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] \qquad ...(6)$$

This is same as the gravitational force between sun and planet given by

$$F = -\frac{GMm}{r^2} \qquad ...(7)$$

Equation (6) and (7), we obtain

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2} \qquad ...(8)$$

Let us put $r = \frac{1}{u}$, so that
$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}(hu^2) \qquad \left(\because \frac{d\theta}{dt} = \frac{h}{r^2} = hu^2\right)$$

$$\frac{dr}{dt} = -h\frac{du}{d\theta}$$

Differentiating again, we get
$$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\cdot\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}$$

Putting various values in equation (8), we get

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -GMu^2$$

or
$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}$$

or
$$\frac{d^2u}{d\theta^2} + \left(u - \frac{GM}{h^2}\right) = 0 \qquad ...(9)$$

Since $\frac{GM}{h^2}$ is a constant, we may write

$$\frac{d^2}{d\theta^2}\left(u-\frac{GM}{h^2}\right)+\left(u-\frac{GM}{h^2}\right)=0$$

or
$$\frac{d^2y}{d\theta^2}+y=0 \qquad \text{...(10)}; \quad y=u-\frac{GM}{h^2}$$

The solution of differential equation (10) is of the form

$$y=u-\frac{GM}{h^2}=A\cos\theta, \qquad A \text{ is a constant}$$

or
$$u=\frac{GM}{h^2}+A\cos\theta$$

or
$$\frac{1}{r}=\frac{GM}{h^2}+A\cos\theta$$

Dividing throughout by $\frac{GM}{h^2}$, we get

$$\frac{h^2/GM}{r}=1+\frac{h^2A}{GM}\cos\theta \qquad \text{...(11)}$$

This equation is of the form
$$\frac{l}{r}=1+e\cos\theta \qquad \text{...(12)}$$

which represents a conic section of ecentricity $e=\frac{h^2A}{GM}$ and semi latus-rectum $l=h^2/GM$. Since the orbit of the planet around the sun must be a closed one (*i.e.*, the planet do not escape from sun's gravitational field), the total energy of the planet $E=\frac{1}{2}mv^2-\frac{GMm}{r}$ should be negative.

Since,
$$e=\sqrt{1+\frac{2Eh^2}{M^2G^2}}$$

It may be seen that E is negative only when $e<1$.

Now for $e<1$, conic section represented by equation (12) will be an ellipse. Hence the orbit of the planet around the sun is an ellipse. This is Kepler's first law.

Deduction of Third Law

If a and b be the semi-major and semi-minor axes of the ellipse, then

$$l=\frac{b^2}{a}=\frac{h^2}{GM} \qquad \text{...(13)}$$

If T be the period of revolution of the planet around the sun, then

$$T=\frac{\text{area of ellipse}}{\text{area velocity}}=\frac{\pi ab}{h/2}$$

or
$$T^2 = \frac{4\pi^2 a^2 b^2}{h^2} = \frac{4\pi^2 a^2 b^2}{GMb^2 / a}$$
$$= \frac{4\pi^2}{GM} a^3$$
or
$$T^2 \propto \alpha^3$$

This is Kepler's third law.

Q.5. What are satellites? Discuss the mechanism of launching of artificial satellites.

Ans.

Satellites

There are certain heavenly bodies which revolve round the planets. These bodies are called natural satellites. For example, moon revolves round the earth. Thus moon is a natural satellite of earth. Similarly, other planets have various satellites revolving around them. Ganymade (the largest and most massive of the moons) is a planet of Jupiter.

Now a days it is possible to put certain man-made objects into stable orbits around the earth for some specific purposes. Such objects are called artificial satellites of the earth. Indian National Satellites (INSATS), Aryabhatta, Rohini, Metset, GSAT-30 etc. are some of the artificial satellites launched by India.

Launching of Artificial Satellites

Artificial satellites are launched with the help of multi-stage rockets. Places on earth near the equator are best suited for the launching of satellites due to lesser gravitational attraction in these regions. The satellite is placed upon the rocket which is launched from the earth. After the rocket reaches its maximum vertical height h, a special mechanism provides a thrust to the satellite at point P (fig. 1). This gives a horizontal velocity v to the satellite.

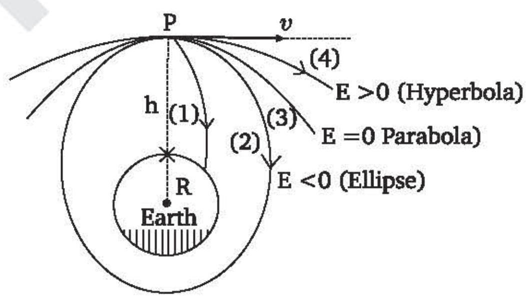


Fig. 1

The total energy of the satellite at point P is
$$E = KE + PE = \frac{1}{2}mv^2 - \frac{GMm}{r} \quad ...(1), \; r = R + h$$

where m is the mass of the satellite and M and R are the mass and radius of the earth. The shape of the orbit of the satellite round the earth depends upon the total energy of the satellite as follows :

$E = 0$ parabolic

$E < 0$ ellipse

$E > 0$ hyperbola

In every case, the centre of the earth is one focus of the orbit. For small negative energy values, the elliptical orbit will intersect the earth (curve (1)) and the satellite will fall back on the earth. For other energy values the satellite will continue to move in a closed orbit, or will escape for the earth's gravitational field depending on the values of v and r.

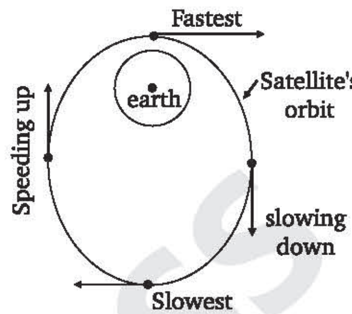


Fig. 2

A satellite which is launched horizontally with a speed of 8 km/s such that $h << R$ will be bent into a circular orbit by the earth's gravitational attraction. However, if it is launched horizontally with a speed $v < 8$ km/s, then it will fall back on the earth. Thus 8 km/s is the critical (minimum) speed for the satellite to move in a closed orbit.

The satellite follows an elliptical path $(E < 0)$ around the earth if it is launched horizontally with a speed that is greater than 8 km/s but less than 11 km/s. In the elliptical path the speed of the satellite varies continuously. Its speed is maximum at the nearest point to the earth and minimum at the farthest point as shown in fig. 2. The nearest point is known as **perihelion** whereas the farthest point is called **aphelion**.

For speeds greater than 11 km/s, the **energy of satellite** as given by equation (1) will become **positive** and the satellite is **no longer bound** to the earth. It escapes from the earth's gravitational field following a hyperbolic path. When satellite is in a circular orbit of radius r, then

$$\frac{mv^2}{r} = \frac{GMm}{r^2} \Rightarrow \frac{1}{2}mv^2 = \frac{GMm}{2r}$$

$\therefore$
$$E = \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$$

Q.6. A satellite of mass m is going round the earth (mass M_e) in a circular orbit of radius R_e. Write down its angular momentum J about the centre of its orbit and express its total energy E in terms of its angular momentum.

Sol. Angular momentum of a satellite of mass m about the centre of its orbit is given by

$$\vec{J} = \vec{r} \times m\vec{v}$$

or
$$|\vec{J}| = J = mvr \sin\theta$$

If the satellite is moving in a circular orbit $(\vec{r} \perp \vec{v})$ then, $\theta = 90°$

$\therefore$
$$\sin\theta = 1$$

Hence
$$J = mvr$$

Also
$$\frac{mv^2}{r} = \frac{GM_em}{r^2}$$

$\therefore$
$$mv^2r = GM_em$$

or
$$m^2v^2r^2 = GM_em^2r$$

or
$$J = mvr = (GM_em^2r)^{1/2}$$

Given that radius of orbit $r = R_e$, therefore required angular momentum

$$J = (GM_e \cdot m^2R_e)^{1/2} \qquad \text{...(1)}$$

Kinetic energy of the satellite, $K = \frac{1}{2} mv^2$

and $J^2 = m^2 v^2 r^2 = \frac{1}{2} mv^2 (2mr^2) = K \times 2mr^2$

$\therefore$ $K = \frac{J^2}{2mr^2}$ ...(2)

Also potential energy, $U = -\frac{GM_e m}{r}$

Since $\frac{GM_e m}{r^2} = \frac{mv^2}{r} \Rightarrow \frac{GM_e m}{r} = mv^2$

$\therefore$ $U = -mv^2$

Now $J^2 = m^2 v^2 r^2 = -mv^2 (-mr^2) = -mr^2 U$

$\therefore$ $U = -\frac{J^2}{mr^2}$ ...(3)

$\therefore$ Total energy of the satellite $E = K + U = \frac{J^2}{2mr^2} - \frac{J^2}{mr^2} = -\frac{J^2}{2mr^2}$

Since, $r = R_e$, therefore $E = -\frac{J^2}{2mR_e^2}$ ...(4)

is the required relation between total energy and angular momentum.

Q.7. Show that for a satellite :

(i) Period of revolution, $T = \frac{2\pi (R+h)^{3/2}}{R\sqrt{g}}$.

(ii) Total energy, $E` = -\frac{GMm}{2r}$.

Sol. (i) **Period of Revolution of a Satellite :** The time taken by the satellite to complete one round about the earth (planet) is called its period of revolution. Let it be denoted by T. Then

$$T = \frac{\text{circumference of the circular orbit}}{\text{orbital speed}}$$

$$= \frac{2\pi r}{v} = \frac{2\pi (R+h)}{v}, \qquad \because r = R + h$$

$$= \frac{2\pi (R+h)}{R\sqrt{g/(R+h)}}$$

or $$T = \frac{2\pi (R+h)^{3/2}}{R\sqrt{g}}$$

The period of revolution of a satellite increases with increase in the radius of the orbit.
(ii) **Energy of a Satellite :** Let a satellite of mass m is revolving around the earth in a circular orbit of radius r. Then potential energy of the system will be

$$U(r) = U(\infty) - W_{\infty r}$$

where $U(\infty)$ is the potential energy of the satellite when it is at infinite distance away from the earth and $W_{\infty r}$ is the work done by the gravitational force to move the satellite from infinity to a distance r. The earth being much heavier than satellite, is assumed to be at rest $(v = 0)$. Since $U(\infty)$ can be taken to be zero, therefore

$$U(r) = -W_{\infty r} = -\int_{\infty}^{r} F(r)\,dr \qquad ...(1)$$

But gravitational force of the earth on satellite at a distance r from earth is

$$F(r) = -\frac{GMm}{r^2}, \; M = \text{mass of earth}$$

$$\therefore \qquad U(r) = -\int_{\infty}^{r} \frac{GMm}{r^2}\,dr = -\frac{GMm}{r} \qquad ...(2)$$

So the potential energy of the satellite is negative.
Kinetic energy of the system = KE of earth + KE of the satellite

$$K = 0 + \frac{1}{2}mv^2$$

where v is the orbital speed of the satellite.

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

or

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2r} \qquad ...(3)$$

The kinetic energy is always positive. The total energy of the system is

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r}$$

or

$$E = -\frac{GMm}{2r} \qquad ...(4)$$

Thus the total energy is constant and negative. The negative energy signifies that the satellite is always bound to the earth by the gravitational attraction and moves in a closed orbit. Even in the elliptical orbits in which r and v vary, the total energy is constant and negative although KE and PE may vary individually.

❑

UNIT-VIII

Wave Motion

SECTION-A VERY SHORT ANSWER TYPE QUESTIONS

Q.1. Define periodic motion.

Ans. Periodic Motion : Any motion of a system that is continuously and identically repeated. The time *T* that it takes to complete one cycle of an oscillation or wave motion is called the period, which is the reciprocal of the frequency. *e.g.*, pendulum, simple harmonic motion.

Q.2. What is meant by amplitude?

Ans. Amplitude : The maximum displacement or distance moved by a point on a vibrating body or move measured from its equilibrium position. It is equal to one-half the length of the vibration path. The amplitude of a pendulum is thus one-half the distance that the bob traverse in moving from on side to the other. Waves are generated by vibrating sources, their amplitude being proportional to the amplitude being proportional to the amplitude of the source.

Q.3. What do you mean by restoring force?

Ans. The restoring force is a force which acts to bring a body to its equilibrium position. The restoring force is a function only of position of the mass or particle and it is always directed back toward the equilibrium position of the system. The restoring force is often referred to in simple harmonic motion.

Q.4. Write very short note on simple harmonic motion.

Ans. Simple harmonic motion : Repetitive movement back and forth through an equilibrium or central position, so that the maximum displacement on one side of this position is equal to the maximum displacement on the other side. The time interval of each complete vibration is the same.

Q.5. What are damped oscillations?

Ans. The effect of radiation by an oscillating system and of the friction present in the system is that amplitude of oscillations gradually diminishes with time. The reduction in amplitude (or energy) of an oscillator is called damping and the oscillation are said to be damped.

Q.6. Define velocity resonance. What is the difference between amplitude and velocity resonance?

Ans. Velocity resonance : This is the phase difference at which maximum energy is transferred from the applied oscillator to the resonating oscillator.

The difference is that in case of amplitude resonance the energy of the forced vibration may not be maximum but in case of velocity resonance the energy is always maximum.

Q.7. Write about logarithemic decrement in brief.

Ans. The logarithmic decrement represents the rate at which the amplitude of a free damped vibration decrease. It is defined as the natural logarithm of the ratio of any two successive amplitudes. It is found from the time response of underdamped vibration (oscilloscope or real time analyzer).

Q.8. What is meant by wave motion?

Ans. Wave motion (propagation of disturbance), that is deviations from a state of rest or equilibrium from place to place in a regular and organised way. Most familiar are surface waves on water, but both sound and light travel as wave like disturbances and the motion of all subatomic particle exhibits wave like properties.

Q.9. What is phase velocity and wave velocity?

Ans. Waves can be in the group and such groups are called wave packets, so the velocity with a wave packet travels is called group velocity. The velocity with the phase of a wave travels is called phase velocity. The relation between group velocity and velocity are proportionate.

Q.10. What do you understand by intensity of wave.

Ans. Intensity is defined to be the power per unit are a carried by a wave. Power is the rate at which energy is transferred by the wave. In equation form, intensity I is $I`= PA$.
Where, P is the power through an area A.

SECTION-B SHORT ANSWER TYPE QUESTIONS

Q.1. What is compound pendulum? Explain it with equation.

Ans. Compound Pendulum : A rigid body capable of oscillating freely in a vertical plane about a horizontal axis passing through it is called a compound pendulum. The centre of suspension of the pendulum is the point in which the axis of rotation meets the vertical plane through the centre of gravity of pendulum.

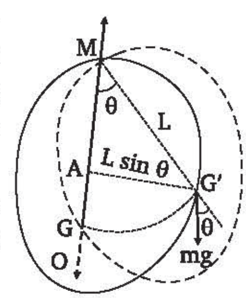


Fig. 1

Fig. 1 represents the vertical section of a rigid body capable of oscillations about a horizontal axis through the centre of suspension M. Let G be the centre of gravity of the body with $MG = L$.

Now, if the body is displaced through an angle θ, its centre of gravity moves to G' and body starts oscillating. The weight (mg) of the body and its reaction at the support constitute a couple, given by

$$C = -mg\,(G'A) = -mg\,L\sin\theta$$

This is the restoring couple as it tends to bring the displaced body to its original position. If I be the moment of inertia of the body about the horizontal axis through M, then the couple is $I\dfrac{d^2\theta}{dt^2}$.

Therefore,

$$I\frac{d^2\theta}{dt^2} = -mgL\sin\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{m}{I} g\, L \sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} \left(\frac{mgL}{I}\right)\theta = 0 \qquad [\because \sin \approx \theta \text{ if } \theta \text{ is small}]$$

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \qquad ...(1)$$

where $\omega = \sqrt{\frac{mgL}{I}}$, is the angular frequency of oscillations. Equation (1) represents that the pendulum is executing a SHM, whose periodic time is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgL}} \qquad ...(2)$$

Solution of eqn. (1) is $\theta = \theta_0 \sin(\omega t + \phi)$ ...(3)

If I_g be the M.I. of the body about a parallel axis passing through its centre of gravity, then from the theorem of parallel axes

$$I = I_g + mL^2$$

or

$$I = mK^2 + mL^2$$

where K is the radius of gyration of the body about a parallel axis passing through the centre of gravity.

$$T = 2\pi\sqrt{\frac{m(K^2 + L^2)}{mgL}} = 2\pi\sqrt{\left(\frac{K^2}{L} + L\right)/g} \qquad ...(4)$$

Thus period of oscillation is same as that of a simple pendulum of length $\left(\frac{K^2}{L} + L\right)$. This length is called the length of an equivalent simple pendulum.

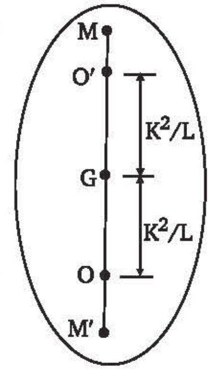


Fig. 2

If we take a point 'O' on the line MG produced such that $MO = \left(\frac{K^2}{L} + L\right)$, then point O is called the centre of oscillation.

Q.2. Define the following : (i) Relaxation time, (ii) Quality factor of a damped oscillator.

Ans.

(i) Relaxation Time

The time after which the total energy of the damped harmonic oscillator remains $(1/e)$ times its initial value is called relaxation time. Now at any time t, the total energy of the oscillator is

$$E = E_0 e^{-2rt}$$

If τ is the relaxation time, then

$$E_\tau = \frac{E_0}{e}$$

or $$E_0 e^{-2rt} = \frac{1}{e} \cdot E_0$$

or $$e^{-2rt} = e^{-1}$$

or $$\tau = \frac{1}{2r}$$

(ii) Quality Factor of a Damped Oscillator

The quality factor is given by

$$Q = 2\pi \cdot \frac{\text{Energy stored in the oscillator}}{\text{Energy dissipated in one time period}}$$

$$= 2\pi \cdot \frac{E}{PT}$$

where P is the average loss of energy over a period $= E/\tau$

Since $$Q = 2\pi \cdot \frac{E}{2rE \cdot (2\pi/\omega)} = \frac{\omega}{2r}$$

or $$Q = \omega\tau \qquad \because \quad \tau = \frac{1}{2r}$$

where τ is the relaxation time.

Q.3. Explain the velocity of forced harmonic oscillator.

Ans.

Velocity of Forced Harmonic Oscillator

The displacement of forced harmonic oscillator is :

$$x = \frac{f_0}{[(\omega_0^2 - p^2)^2 + 4r^2p^2]^{1/2}} \sin(pt - \phi) \quad \text{...(1)}$$

Therefore, the velocity of the oscillator is given by

$$v = \frac{dx}{dt} = \frac{f_0}{[(\omega_0^2 - p^2)^2 + 4r^2p^2]^{1/2}} \cdot p\cos(pt - \phi)$$

or $$v = \frac{f_0 p}{[(\omega_0^2 - p^2)^2 + 4r^2p^2]^{1/2}} \sin\left[(pt - \phi) + \frac{\pi}{2}\right]$$

or $$v = v_0 \sin\left[(pt - \phi) + \frac{\pi}{2}\right] \quad \text{...(2)}$$

where, $$v_0 = \frac{f_0 p}{[(\omega_0^2 - p^2)^2 + 4r^2p^2]^{1/2}} \quad \text{...(3)},$$ is the velocity amplitude.

From equation (2), it is clear that velocity leads the displacement (or driving force) by $\frac{\pi}{2}$.

Equation (3) may be written as

$$v_0 = \frac{f_0}{\left[\left(\frac{\omega_0^2 - p^2}{p}\right)^2 + 4r^2\right]^{1/2}}$$

For $p \approx \omega_0$, the velocity amplitude is maximum at a particular frequency. This occurs when the denominator in v_0 becomes minimum, for which

$$\left(\frac{\omega_0^2 - p^2}{p}\right) = 0$$

or $$p = \omega_0$$

Thus for every value for damping, the velocity amplitude is maximum when the frequency of applied force is equal to the natural frequency of the oscillator. The phenomenon of velocity becoming maximum, is known as "velocity resonance".

Q.4. The energy of a damped harmonic oscillator is reduced to one-eight of its original value in 25 seconds. Find the damping constant.

Sol. The energy of a dampled harmonic oscillator is given by

$$E = E_0 e^{-2rt} = E_0 e^{-t/\tau}$$

where E_0 is the initial value of energy, τ is relaxation time and E is the energy remained after time t.

Given, $$E = E_0/8 \text{ at } t = 25 \text{ seconds}$$

$\therefore$ $$\frac{E_0}{8} = E_0 e^{-25/\tau} \quad \Rightarrow \quad \frac{1}{8} = e^{-25/\tau}$$

or $$\frac{25}{\tau} = \log_e 8 = 2.3 \log_{10} 8 = 2.3 \log_{10} 2^3$$

$$= 6.9 \log_{10} 2 = 6.9 \times 0.3010$$

$$\frac{25}{\tau} = 2.0769$$

or $$\tau = \frac{25}{2.0769} = 12.04 \text{ s}$$

Thus, relaxation time will be 12.04 s.

Q.5. Deduce the differential equation of wave motion.

Ans.

Differential Equation of Wave Motion

Equation of motion of a particle at a distance x from the origin is given by

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \qquad ...(1)$$

where a is amplitude, y the displacement of particle at any time t and v is the velocity of wave.

Differentiating equation (1) with respect to t, keeping x constant, we get the velocity of the particle

$$u = \frac{dy}{dt} = \frac{2\pi\, va}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \qquad \text{...(2)}$$

Differentiating again with respect to t, the acceleration of the particle

$$f = \frac{d^2 y}{dt^2} = -\frac{4\pi^2 y^2}{\lambda^2} a \sin \frac{2\pi}{\lambda}(vt - x) \qquad \text{...(3)}$$

Differentiating equation (1) with respect to x, the strain is given by

$$\frac{dy}{dx} = \frac{-2\pi a}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \qquad \text{...(4)}$$

Differentiating equation (4) with respect to x, the rate of change of strain with distance is given by

$$\frac{d^2 y}{dx^2} = -\frac{4\pi^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x) \qquad \text{...(5)}$$

Now, from equation (2) and (4), we get

$$\frac{dy}{dt} = -v \frac{dy}{dx}$$

or

$$u = -v \frac{dy}{dx} \qquad \text{...(6)}$$

i.e., Particle velocity = – Wave velocity × Slope of displacement curve

Now, comparing (3) and (5), we get

$$\frac{d^2 y}{dt^2} = v^2 \frac{d^2 y}{dx^2} \qquad \text{...(7)}$$

This is the differential equation of wave motion.

Q.6. Write short note on transverse vibrations of a stretched string.

Ans.

Transverse Vibrations of a Stretched String

Let a string be stretched between points A and B with a tension T. If the string is slightly displaced to one side of the centre and then released, it starts to vibrate at right angles to its length (Fig.).

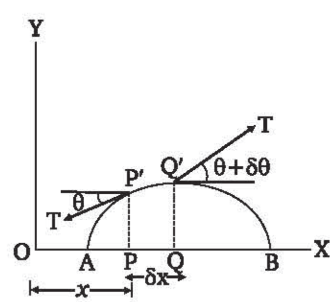


Fig.

Let us take the x-axis along the direction of the undisplaced spring AB and the y-axis in the direction of displacement of the particles at right angles to AB. Let PQ be a small element of the string of the length δx at a distance x from O.

Let $P'Q'$ be the displaced position of the element at any time t, with a displacement y. Since the string is perfectly flexible, the tension will be the same at each point of the string along the tangent at that point.

If θ and $(\theta+\delta\theta)$ be the inclination with x-axis of the tension acting at points P' and Q' respectively. Then we have

$$T_x = T\cos(\theta+\delta\theta) - T\cos\theta$$

$$T_y = T\sin(\theta+\delta\theta) - T\sin\theta$$

Since θ and $\delta\theta$ are small,

$\therefore$ $$\cos(\theta+\delta\theta) \approx \cos\theta, \sin(\theta+\delta\theta) \approx \theta+\delta\theta \text{ and } \sin\theta = \theta$$

$\therefore$ $$T_x = 0 \text{ and } T_y = T(\theta+\delta\theta) - T\theta = T\delta\theta$$

For small angles, $\theta = \tan\theta$

Hence resultant force along Y-axis

$$= T\delta(\tan\theta) = T\delta\left(\frac{\partial y}{\partial x}\right) = T\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\delta x = T\frac{\partial^2 y}{\partial x^2}\delta x$$

If m is the mass per unit length of the string then the mass of small element δx will be $m\delta x$. If the acceleration of this element at the displacement y be $\left(\frac{\partial^2 y}{\partial t^2}\right)$, then according to Newton's second law of motion

$$m\delta x\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}\delta x$$

or $$\frac{\partial^2 y}{\partial t^2} = \left(\frac{T}{m}\right)\frac{\partial^2 y}{\partial x^2} \quad \text{...(1)}$$

This is the differential equation of a vibrating string. Comparing this equation with the differential equation of a wave motion

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}, \text{ we get}$$

$$v^2 = \frac{T}{m}$$

or $$v = \sqrt{\left(\frac{T}{m}\right)} \quad \text{...(2)}$$

This equation gives us the velocity of transverse waves moving along the string.

The differential equation for the vibrating string can always be satisfied by the general solution $y = f\left(t - \frac{x}{v}\right) + g\left(t + \frac{x}{v}\right)$.

The first part represents a wave travelling towards the positive x-axis with speed v and the second part represents a wave travelling toward the negative direction of x-axis.

Q.7. Define the following : (i) Superposition of waves, (ii) stationary waves.

Ans.

Superposition of Waves

Huygens, first proposed the principle of superposition, according to which if two or more independent waves are propagated through a medium, all at the same time, the resultant displacement at any point is the vector sum of the displacements due to each individual wave *i.e.,*

$$y = y_1 + y_2 + y_3 + \dots$$

For the principle of superposition to apply, the equation of the waves must be a linear one. The importance of the principle of superposition lies in the fact that in the case where it holds good, it is possible to analyze a complicated wave motion into a set of simple waves.

Stationary Waves : The superposition of two waves of same period, wavelength and amplitude travelling with same velocity in opposite directions gives rise to standing or stationary waves.

Equation of Stationary Waves

The progressive wave of wavelength λ and amplitude a travelling with velocity v along positive x-axis is given by

$$y_1 = a \sin \frac{2\pi}{\lambda}(vt - x) \quad \text{...(1)}$$

where y_1 is the displacement of a particle at any time t at a distance x from the origin. The reflected wave is travelling in opposite direction with same velocity v, therefore its equation is given by

$$y_2 = \pm a \sin \frac{2\pi}{\lambda}(vt + x) \quad \text{...(2)}$$

The positive and negative signs are used if the boundary of the medium is free and rigid respectively.

Q.8. Write the differences between progressive and stationary waves.

Ans. Differences between Progressive and Stationary Waves are as follows :

	Progressive Waves	Stationary Waves
1.	These forward waves travel in the medium with a finite velocity.	There is no advancement of the waves in either direction.
2.	All points of the medium vibrate with the same amplitude.	The amplitude is zero at the node and maximum at the antinode.
3.	The phase of vibration varies continuously from point to point.	All points between any two consecutive nodes vibrate in the same phase, but the phase suddenly reverse at each node.
4.	No point is permanently at rest. However each point is momentarily at rest at its maximum displacement.	The nodes are permanently at rest. Other points are momentarily at rest at their maximum displacements.
5.	Different points cross their mean positions in succession and have the same velocity.	All points reach the position of maximum displacement simultaneously two-times in each period.

6.	All points cross their mean positions in succession and have the same velocity.	All points cross their mean positions simultaneously but with different velocities. Antinodes have maximum velocity but nodes have zero velocity.
7.	The pressure-variation being same at all points of the medium, travels forward.	The pressure-variation is different at different points. It is maximum at the nodes and minimum at the antinodes. It does not move forward but occurs simultaneously at all places.
8.	There is flow of energy in the direction of propagation of the wave.	There is no flow of energy acorss any plane.

Q.9. A simple harmonic wave travelling along x-axis is given by $y = 5 \sin 2\pi (0.2t - 0.5x)$ meter. Calculate the amplitude, frequency, wavelength, wave velocity, particle velocity and particle acceleration.

Sol. The given equation is

$$y = 5 \sin 2\pi (0.2t - 0.5x) \text{ m}$$

$$= 5 \sin 2\pi \left(\frac{t}{5} - \frac{x}{2} \right) \text{ m} \quad ...(1)$$

The general equation of a simple harmonic wave is

$$y = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda} \right) \quad ...(2)$$

Comparing equation (1) with (2), we get

amplitude of the wave, $a = 5$ m

period of the wave, $T = 5$ s

$\therefore$ frequency of the wave, $n = \frac{1}{T} = \frac{1}{5} = 0.2$ cycle/s

wavelength, $\lambda = 2$ m

wave velocity, $v = n\lambda = 0.2 \times 2 = 0.4 \text{ m/s} = 40 \text{ cm/s}$

Differentiating equation (1) w.r.t. time t, we get

$$\text{particle velocity} = \frac{\partial y}{\partial t} = 5 \times \frac{2\pi}{5} \cos 2\pi \left(\frac{t}{5} - \frac{x}{2} \right) \text{ m/s}$$

$$= 2\pi \cos 2\pi \left(\frac{t}{5} - \frac{x}{2} \right) \text{ m/s}$$

$$\text{particle acceleration} = \frac{\partial^2 y}{\partial t^2} = -2\pi \times \frac{2\pi}{5} \sin 2\pi \left(\frac{t}{5} - \frac{x}{2} \right) \text{ m/s}^2$$

$$= \frac{-4\pi^2}{5} \sin 2\pi \left(\frac{t}{5} - \frac{x}{2} \right) \text{ m/s}^2$$

Q.10. What are Lissajou's figures? Describe the composition of two mutually perpendicular SHM of same frequency.

Ans.

Lissajou's Figures

When two simple harmonic motions at right angles to each other act on a particle simultaneously, the resultant path of the particle, in general be, a closed curve and is called a Lissajous figure. The nature of the curve traced out depends on :

(i) component amplitudes,

(ii) ratio of frequencies or periods, and

(iii) relative phase of component motions.

Composition of two Mutually Capserpendicular SHM of same Frequency

Let us consider a particle acted on by two SHM's of the same period or frequency and of different amplitude and phase. Let one SHM is along x-axis and the other along y-axis. These are :

$$x = a\sin(\omega t + \phi) \qquad ...(1)$$

and

$$y = b\sin\omega t \qquad ...(2)$$

where a and b are respective amplitudes and ϕ is the phase difference between them. $T = \frac{2\pi}{\omega}$ is the period of both the motions.

From equation (1), we have

$$\frac{x}{a} = \sin\omega t \cdot \cos\phi + \cos\omega t \cdot \sin\phi$$

or

$$\frac{x}{a} = \frac{y}{b}\cos\phi + \sqrt{\left(1 - \frac{y^2}{b^2}\right)}\sin\phi$$

or

$$\frac{x}{a} - \frac{y}{b}\cos\phi = \sqrt{\left(1 - \frac{y^2}{b^2}\right)}\sin\phi$$

On squaring both sides, we get

$$\left(\frac{x}{a} - \frac{y}{b}\cos\phi\right)^2 = \left(1 - \frac{y^2}{b^2}\right)\sin^2\phi$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}\cos^2\phi - \frac{2xy}{ab}\cos\phi = \sin^2\phi - \frac{y^2}{b^2}\sin^2\phi$$

or

$$\frac{x^2}{a^2}(\cos^2\phi + \sin^2\phi) - \frac{2xy}{ab}\cos\phi = \sin^2\phi$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\phi = \sin^2\phi \qquad ...(3)$$

This equation represents an **oblique ellipse**, which is the resultant path of the particle.

SECTION-C LONG ANSWER TYPE QUESTIONS

Q.1. Write about springs and mass system in detail.

Ans.

Spring and Mass System

The system consists of a massless spring, one end of which is connected to a mass m while the other end of the spring is connected to a fixed point. Two positions are possible :

1. **System is on a Horizontal Surface :**

(i) **Massless spring :** Let the mass and the spring are on a smooth horizontal surface (Fig. 1). Let us consider that a force F is applied on the spring to stretch or compress it, then the spring exerts an equal and opposite force on the mass.

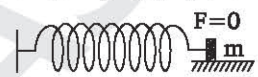


Fig. 1

The restoring force exerted by the spring is given by

$$F = -Cx$$

where C is the force constant and x is the extension or compression produced in the spring.

If $\frac{d^2x}{dt^2}$ is the acceptance of mass m, then

$$m\frac{d^2x}{dt^2} = -Cx$$

or

$$\frac{d^2x}{dt^2} + \frac{C}{m}x = 0$$

i.e.,

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \quad \text{...(1); where } \omega_0^2 = \frac{C}{m}$$

This is the differential equation of mass m attached to a spring. The above equation shows that the motion of mass-spring system is simple harmonic whose time-period is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{(m/C)}$$

The solution of equation (1) is given by

$$x = a \sin(\omega_0 t + \phi)$$

where ϕ is phase-difference.

(ii) **Spring of Finite Mass :** Let the spring be of finite mass m_S such that $m_S << m$. The spring will stretch uniformly along its length. Let l be the length of the spring.

Mass per unit length of the spring $= \frac{m_S}{l}$.

We consider a small element of the spring of length dS at a distance S from the fixed end.

$$\text{Mass of the element} = \frac{m_S}{l}\, dS$$

When the displacement of the mass m is x, the displacement of the element dS will be $\left(\frac{S}{l}\right)x$ and its velocity will be $\left(\frac{S}{l}\right)\frac{dx}{dt}$.

Instantaneous kinetic energy of the element

$$=\frac{1}{2}\left(\frac{m_S}{l}dS\right)\left(\frac{S}{l}\frac{dx}{dt}\right)^2=\frac{m_S}{2l^3}\left(\frac{dx}{dt}\right)^2 D^2 dS$$

Instantaneous KE of the whole spring

$$=\frac{m_S}{2l^3}\left(\frac{dx}{dt}\right)^2\int_0^l S^2 dS=\frac{1}{6}m_S\left(\frac{dx}{dt}\right)^2$$

Total kinetic energy of the system (spring and mass) is

$$K=\frac{1}{2}m\left(\frac{dx}{dt}\right)^2+\frac{1}{6}m_S\left(\frac{dx}{dt}\right)^2=\frac{1}{2}\left(m+\frac{m_S}{3}\right)\left(\frac{dx}{dt}\right)^2$$

Clearly, effective mass of the system $=m+\frac{m_S}{3}$

Time period of the system, $T=2\pi\sqrt{\left(m+\frac{m_S}{3}\right)/C}$

2. System is in Vertical Position : When mass m is attached to the spring in vertical position (Fig. 2), then weight mg of the body produces an elongation x_0 in its initial length.
In this condition $mg=Cx_0$ the body is in equilibrium position.
If x is the displacement of m from equilibrium, then total restorting force will be

$$F=-C\,(x_0+x)$$

Hence effective restoring force $=mg-Cx$

$$=Cx_0-Cx=-C\,(x-x_0$$

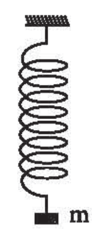


Fig. 2

Equation of motion in this case is

$$\frac{d^2x}{dt^2}+\frac{C}{m}x=0$$

or

$$\frac{d^2x}{dt^2}+\omega_0^2x=0,\ \text{ where } \omega_0^2=(C/m)$$

This is the differential equation of simple harmonic motion. Its time period

$$T=\frac{2\pi}{\omega}=2\pi\sqrt{m/C}$$

i.e., gravity has no effect on oscillations because time period is same as in the case of horizontal system.

Q.2. Define power dissipation in damped harmonic oscillator.

Ans. Power Dissipation in Damped Harmonic Oscillator

If a oscillator oscillates in a medium, then its amplitude of oscillation decreases due to frictional force produced by factors present in the medium or in the oscillator itself. This type of oscillator is called damped oscillator. The amplitude is continuously decreasing because the energy of oscillator is used in the form of heat in doing work against the friction forces. This is called power dissipation.

At any time t, the displacement of a damped harmonic oscillator is given by

$$x = ae^{-rt} \sin(\beta t + \phi) \qquad ...(1)$$

Here a and ϕ are arbitrary constants, r is damping constant and $\beta = \sqrt{(\omega^2 - r^2)}$ is the angular frequency of damped oscillator and $\omega = \sqrt{(k/m)}$ is the frequency of undamped oscillator, and r is force constant.

For low damping, $r^2 < \omega^2$, then

$$x = ae^{-rt} \sin(\beta t + \phi)$$

$$\frac{dx}{dt} = -are^{-rt} \sin(\beta t + \phi) + ae^{-rt} \cdot \beta \cdot \cos(\beta t + \phi)$$

$$= ae^{-rt} [\beta \cos \beta t - r \sin \beta t]$$

where we have assumed initial phase $\phi = 0$.

The K.E. of particle at any time t, is given by

$$K = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2$$

$$= \frac{1}{2} ma^2 e^{-2rt} (\beta \cos \beta t - r \sin \beta t)^2$$

$$= \frac{1}{2} ma^2 e^{-2rt} (\beta^2 \cos^2 \beta t + r^2 \sin^2 \beta t - 2\beta r \sin \beta t \cos \beta t)$$

or

$$K = \frac{1}{2} ma^2 e^{-2rt} (\beta^2 \cos^2 \beta t + r^2 \sin^2 \beta t - r\beta \sin 2\beta t) \qquad ...(2)$$

Average kinetic energy for a complete time period

$$\overline{K} = \frac{1}{T} \int_0^T K \, dt$$

$$= \frac{1}{T} \int_0^T \frac{ma^2 e^{-2rt}}{2} [\beta^2 \cos^2 \beta t + r^2 \sin^2 \beta t - r\beta \sin 2\beta t] dt$$

Bet for a complete time period the average value of $\sin^2 \beta t$ or $\cos^2 \beta t$ is $1/2$ and that of $\sin 2\beta t$ is zero. Hence

$$\overline{K} = \frac{ma^2 e^{-2rt}}{2} \left[\frac{\beta^2}{2} + \frac{r^2}{2} - \beta r \times 0\right]$$

$$=\frac{ma^2e^{-2rt}}{4}[(\omega^2-r^2)+r^2] \qquad \because \beta=\sqrt{\omega^2-r^2}$$

$$=\frac{1}{4}ma^2\omega^2e^{-2rt}$$

Potential energy of the damped oscillator is given by

$$U=\frac{1}{2}kx^2=\frac{1}{2}k(ae^{-rt}\sin\beta t)^2$$

$$=\frac{1}{2}ka^2e^{-2rt}\sin^2\beta t$$

Therefore average potential energy for a complete time period is

$$\overline{U}=\frac{1}{2}\int_0^T\frac{1}{2}m\omega^2a^2e^{-2rt}\sin^2\beta t\,dt, \qquad k=m\omega^2$$

$$=\frac{1}{2}m\omega^2a^2e^{-2rt}\left[\int_0^T\frac{\sin^2\beta t\,dt}{T}\right]$$

$$=\frac{1}{2}m\omega^2a^2e^{-2rt}\times\frac{1}{2}$$

or
$$\overline{U}=\frac{1}{4}m\omega^2a^2e^{-2rt}$$

Hence for a complete time period total average energy of the damped oscillator is given by

$$\overline{K}=\overline{K}+\overline{U}=\frac{1}{4}m\omega^2a^2e^{-2rt}+\frac{1}{4}m\omega^2a^2e^{-2rt}$$

$$\overline{E}=\frac{1}{2}m\omega^2a^2e^{-2rt}$$

At time $t=0$, the energy of damped oscillator $(=\overline{E}_0)$ will be maximum and then it will go on decreasing with time. Hence,

$$\overline{E}_0=\frac{1}{2}m\omega^2a^2$$

Therefore,
$$\overline{E}=\overline{E}_0e^{-2rt}$$

Hence, power dissipation, $P=\frac{d\overline{E}}{dt}$

$$=-2r\overline{E}_0e^{-2rt}=2r\overline{E}=2rE \quad (\text{writing } E \text{ for } \overline{E}).$$

Q.3. Define simple harmonic motion (SHM) with example. Establish the differential equation of SHM and hence solve it.

Sol.

Simple Harmonic Motion (SHM)

A particle is said to execute simple harmonic motion when it vibrates in such a manner that at any instant the resorting force acting on it is proportional to its displacement from a fixed point in its path and is always directed towards that point. A system executing SHM is called simple harmonic oscillator.

Let us consider a particle of mass m, executing simple harmonic motion. If at any instant the particle is at distance x from the mean point, then velocity and acceleration of the particle are respectively given by

$$v = \frac{dx}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Since the particle is in SHM, hence the restoring force $F \propto -x$

i.e., $$F = -kx \qquad \text{...(1)}$$

where k is a proportionality constant and is called the force constant.

From Newton's law

$$\text{Force} = \text{mass} \times \text{acceleration} \qquad \text{...(2)}$$

i.e., $$-kx = m\frac{d^2x}{dt^2}$$

or $$m\frac{d^2x}{dt^2} + kx = 0$$

or $$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

or $$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \text{...(3)}$$

where $$\omega^2 = \frac{k}{m}$$

Eqn. (3) is the differential equation of simple harmonic motion.

Let the solution of differential eqn. of SHM (*i.e.*, eqn. 3) be

$$x = ae^{\alpha t}$$

Then $$\frac{d^2x}{dt^2} = a\alpha^2 e^{\alpha t}$$

Putting the values in eqn. (3), we obtain

$$a\alpha^2 e^{\alpha t} + \omega^2 ae^{\alpha t} = 0$$

or $$ae^{\alpha t}(\alpha^2 + \omega^2) = 0$$

or $$\alpha^2 + \omega^2 = 0 \text{ because } ae^{\alpha t} = x \neq 0$$

Thus $$\alpha = \pm i\omega, \text{ where } i = \sqrt{-1}$$

The two possible solution of eqn. (3) are

$$x = ae^{-i\omega t} \text{ and } x = ae^{-i\omega t}$$

The most general solution of eqn. (3) will be

$$x = a_1 e^{i\omega t} + a_2 e^{-i\omega t} \qquad \text{...(4)}$$

where a_1 and a_2 are constants.

Eq. (4) may also be written as

$$x = a_1(\cos\omega t + i\sin\omega t) + a_2(\cos\omega t - i\sin\omega t)$$
$$= (a_1 + a_2)\cos\omega t + (a_1 - ia_2)\sin\omega t$$
$$= a\sin\phi\cos\omega t + (a_1 - ia_2)\sin\omega t$$
$$= a\sin\phi\cos\omega t + a\cos\phi\sin\omega t$$

where $a_1 + a_2 = a\sin\phi$ and $a_1 - ia_2 = a\cos\phi$.

Then $$x = a\sin(\omega t + \phi) \qquad \text{...(5)}$$

where a is the maximum value of displacement x, and is called the amplitude of oscillation and constant ϕ is called the initial phase or phase constant.

Eqn. (5) may also be equivalently expressed as

$$x = a\cos(\omega t + \phi) \qquad \text{...(6)}$$

Equations (4), (5), (6) represent different forms of the solution of differential equation of SHM. All these solutions are equivalent to each other.

Alternative solution of eq. (3) :

Multiplying both sides of eqn. (3) by $2\frac{dx}{dt}$, we get

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} + \omega^2\cdot 2x\frac{dx}{dt} = 0$$

On integrating with respect to t, we get

$$\left(\frac{dx}{dt}\right)^2 + \omega^2x^2 = A \qquad \text{...(4)}$$

where A, is the constant of integration

When the displacement is maximum *i.e.*, $x = a$, velocity is zero .*e.*,

$$\frac{dx}{dt} = 0$$

Hence $$0 + \omega^2a^2 = A$$

or $$A = \omega^2a^2$$

Hence, eqn. (4) can be written as

$$\left(\frac{dx}{dt}\right)^2 + \omega^2x^2 = \omega^2a^2$$

or $$\frac{dx}{dt} = \omega\sqrt{a^2 - x^2} \qquad \text{...(5)}$$

This equation gives the velocity of the particle at any time t.

From eqn. (5), we get

$$\frac{dx}{\sqrt{a^2 - x^2}} = \omega\,dt$$

On integrating, we get $\sin^{-1}\left(\frac{x}{a}\right) = \omega t + \phi$

where ϕ is constant of integration,

or $$\frac{x}{a} = \sin(\omega t + \phi)$$

or $$x = a\sin(\omega t + \phi) \qquad ...(6)$$

This is the solution of the differential equation of simple harmonic motion given by eqn. (3). Here a, the maximum value of the displacement, is called the amplitude of oscillation and ϕ is a constant, known as initial phase or phase constant. The term $(\omega t + \phi)$ is called the phase of the vibration.

Time Period : Now, if the time t in eqn. (6) is increased by $\frac{2\pi}{\omega}$, the function becomes

$$x = a\sin\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right]$$

$$= a\sin(\omega t + 2\pi + \phi)$$

$$= a\sin(\omega t + \phi)$$

i.e., the displacement of the particle is the same after a time $(2\pi/\omega)$. Therefore, $(2\pi/\omega)$ is the period (T) of the motion *i.e.*,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

The number of vibrations per second (n) is called the frequency of the oscillator and is given by,

$$n = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$\therefore$ $$\omega = 2\pi n = (2\pi / T)$$

The quantity ω is called the angular frequency of S.H.M.

Exact simple harmonic motions are rare : In simple harmonic motion, restorting force α – (displacement).

In physical systems this linear relationship between force and displacement is not exactly followed. The relationship is linear only under certain limitations. For example, the motion of a simple pendulum is simple harmonic only for vanishingly small amplitude of oscillation. For a spring-mass system, the motion of mass will be simple harmonic only for small amplitudes of oscillation. Diatomic molecule can be treated as a simple harmonic vibrator only for displacements from mean position being small (or potential energy $\frac{1}{2}kx^2$). Such limitations are rarely realized in practice. Hence the exactly simple harmonic motions are rare.

Q.4. What is a bar pendulum? Under what conditions the oscillations of bar pendulum are simple harmonic? Obtain an expression for its time period.

Ans.

Bar-Pendulum

A bar pendulum is a uniform metal bar about one meter long, having holes drilled along its length symmetrically on either side of the centre of gravity. The bar is suspended on a horizontal knife edge, which can be inserted into any desired hole. The bar can be made to oscillate about the knife edge in the vertical plane.

Principle : The determination of g by bar pendulum (a compound pendulum) is based upon the principle of interchangeability of the centres of suspension and oscillation. Two points in the pendulum can be located at equal distance on the opposite sides of its centre of gravity and collinear with it in such a way so that the periods of oscillations about them are equal. One of them will be the centre of suspension and the other will be the corresponding centre of oscillation and the distance L (say) between them will be equal to the length of the equivalent simple pendulum. Then the time period T will be given by

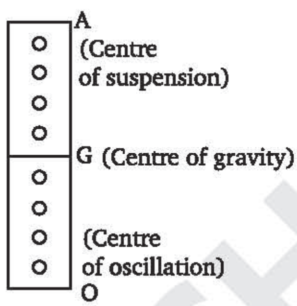


Fig. 1

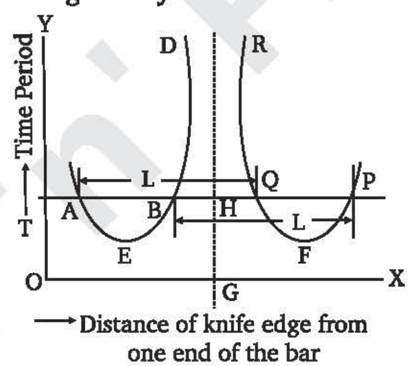


Fig. 2

$$T = 2\pi\sqrt{\left(\frac{L}{g}\right)}$$

Thus if we know T and L, the value of g can be determined.

Method : The time period of different lengths of the pendulum are determined by slipping on to the knife-edge one hole after another from A to G (Fig. 1) and each length carefully noted.

A graph ABD is then plotted between distance of the holes from the centre of gravity along the x-axis and the time period (T) along Y-axis, as shown in fig. 2.

The experiment is repeated with the holes on the other side of the centre of gravity of the pendulum and a similar graph PQR drawn alongside the first on the same scale. This graph PQR will be a mirror image of the first (ABD), as is clear from the fig. 2.

It will be seen at once that as the C.G. of the bar (G) is approached, the time period first decreases, acquires a minimum value and then increases until it becomes infinite at the C.G. itself.

Let a horizontal line AP be drawn, parallel to the x-axis, so as to cut the two curves in point A, B, Q and P.

Then, time period of the pendulum for length corresponding to all these points is the same.

Therefore $AQ = BP = L =$ the length of the equivalent simple pendulum.

$$T = 2\pi\sqrt{(L/g)}$$

or $$g = (4\pi^2 L)/T^2$$

Hence the value of g can be calculated with the help of this equation.

If we draw a tangent, touching the two curves at the points E and F, then at E and F, the centers of suspension and oscillation coincide with each other.

Thus, $$\frac{K^2}{L} = L \text{ or } K = L$$

Hence the distance parallel to x-axis EG or FG is equal to the radius of gyration K.

Thus, $$EF = 2K \quad \text{or} \quad K = \frac{EF}{2}$$

If we note the minimum time period T_m, then

$$T_m = 2\pi\sqrt{\frac{\left(\frac{K^2}{K} + K\right)}{g}} = 2\pi\sqrt{\frac{2K}{g}}$$

From this equation, the value of g can be determined. It is difficult to locate the points E and F, where the tangents to the two curves touches them. Hence K can be determined from the following relation :

$$K^2 = LL' = EG \cdot GR = PG \cdot BG$$

Q.5. Obtain expressions for potential energy, kinetic energy and total energy for a particle executing SHM. Also plot them as a function of displacement from mean position.

Ans.

Energy Considerations in SHM

Let a particle of mass m is undergoing SHM, and at any instant t, particle is at a distance x from its equilibrium position. Then restoring force on the particle is

$$F = -kx$$

where k is force constant.

Potential energy of the particle is given by

$$U = -\int F\,dx = -\int (-kx)\,dx$$

or $$U = \frac{1}{2}kx^2 + C$$

where C, is the constant of integration. If potential energy of the particle at equilibrium position $(x = 0)$ be zero, then

$$U = 0, \text{ at } x = 0, \text{ so } C = 0.$$

P.E. of particle at position x, $$U = \frac{1}{2}kx^2 \qquad ...(1)$$

For SHM, $$x = a\sin(\omega t + \phi)$$

$\therefore$ $$U = \frac{1}{2}ka^2\sin^2(\omega t + \phi) \qquad ...(2)$$

This equation gives the potential energy of a particle at any instant t.

Maximum potential energy, $U_{max} = \frac{1}{2}ka^2$

Kinetic energy of a particle of mass m moving with a velocity v is given by

$$K = \frac{1}{2}mv^2$$

But for SHM,

$$x = a\sin(\omega t + \phi)$$

$\therefore$

$$v = \frac{dx}{dt} = a\omega\cos(\omega t + \phi) \quad \text{...(3)}$$

Hence,

$$K = \frac{1}{2}ma^2\omega^2\cos^2(\omega t + \phi)$$

$$= \frac{1}{2}ka^2\cos^2(\omega t + \phi) \quad \left[\because \omega^2 = \frac{k}{m}\right]$$

$$= \frac{1}{2}ka^2[1 - \sin^2(\omega t + \phi)]$$

$$= \frac{1}{2}k[a^2 - a^2\sin^2(\omega t + \phi)]$$

or

$$K = \frac{1}{2}k(a^2 - x^2) \quad \text{...(4)}$$

This is the kinetic energy of a particle executing SHM.

Maximum kinetic energy, $K_{max} = \frac{1}{2}ka^2$, at $x = 0$.

Therefore, total energy of a particle executing SHM is given by

$$E = U + K$$

$$= \frac{1}{2}ka^2\sin^2(\omega t + \phi) + \frac{1}{2}ka^2\cos^2(\omega t + \phi)$$

$$= \frac{1}{2}ka^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)]$$

or

$$E = \frac{1}{2}ka^2 \quad \text{...(5)}$$

or

$$E = \frac{1}{2}m\omega^2a^2 \quad \left[\because \omega^2 = \frac{k}{m}\right]$$

$$= \frac{1}{2}m(2\pi n)^2a^2$$

$$E = 2\pi^2mn^2a^2 \quad \text{...(6a)}$$

or

$$E = \frac{2\pi^2ma^2}{T^2} \quad \text{...(6b)}$$

Thus, $E \alpha a^2$ and $E \alpha 1/T^2$ or n^2.

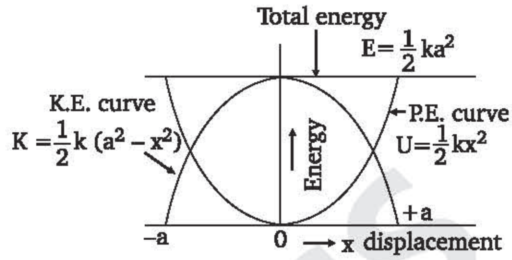


Fig.

Fig. shows the graph between K.E. and displacement (x) of the particle which is the reciprocal of P.E. curve, with the vertex of the parabola touching the upper horizontal line or the total energy curve at $x = 0$. Since the total energy curve is a horizontal line in either case, being parallel to the time-axis in the former and to the displacement axis in the later case, *i.e.,* the total energy of the particle remains constant throughout and is independent of both time and displacement.

Q.6. What do you understand by damped oscillator? Set up the differential equation for damped harmonic motion and solve it to obtain the expression for displacement. Discuss heavy damped critical damped and underdamped cases.

Ans. Damped Harmonic Oscillator : When a damping force acts on a particle executing simple harmonic motion then the amplitude of oscillation does not remain constant but goes on decreasing slowly. If damping is to be taken into account then a harmonic oscillator experiences the following forces :

(a) restoring force $= -kx$; k = force constant.

(b) friction or damping force $= -b\dfrac{dx}{dt}$; b = damping constant.

Hence, total force acting on the particle $= -kx - b\dfrac{dx}{dt}$

Eqn. of motion of damped harmonic oscillator will be

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

or
$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

or
$$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega^2 x = 0 \quad \text{...(1)}$$

where $2r = \dfrac{b}{m}$ = damping for unit mass and unit displacement

and $\omega^2 = \dfrac{k}{m}$ = restoring force for unit mass and unit displacement.

Equation (1) is the **differential equation for damped harmonic oscillator**. Let its solution be

$$y = Ae^{\alpha t} \quad \text{...(2)}$$

where A and α are arbitrary constants

$$\therefore \qquad \frac{dx}{dt} = A\alpha e^{\alpha t} \text{ and } \frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t}$$

Putting these values in eqn. (1), we get

$$A\alpha^2 e^{\alpha t} + 2r\,A\alpha e^{\alpha t} + \omega^2 A e^{\alpha t} = 0$$

or

$$Ae^{\alpha t}[\alpha^2 + 2r\alpha + \omega^2] = 0$$

Since $Ae^{\alpha t} \neq 0$, hence $\alpha^2 + 2r\alpha + \omega^2 = 0$ ...(3)

or

$$\alpha = -r \pm \sqrt{r^2 - \omega^2}$$

Let α_1 and α_2 be the roots of eqn. (3), then

$$\alpha_1 = -r + \sqrt{r^2 - \omega^2}$$

and

$$\alpha_2 = -r - \sqrt{r^2 - \omega^2}$$

Hence general solution of eqn. (1) is given by

$$x = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

or

$$x + A_1 \exp\left(-r + \sqrt{r^2 - \omega^2}\right)t + A_2 \exp\left(-r - \sqrt{r^2 - \omega^2}\right)t \qquad ...(4)$$

where A_1 and A_2 are arbitrary constants. Now for the relative values of k and ω, there are three cases :

I. Heavy damping : $r > \omega$

If $r > \omega$, then $\sqrt{(r^2 - \omega^2)}$ will be real and less than r. Therefore in eqn. (4), $[-r + \sqrt{(r^2 - \omega^2)}]$ and $[-r - \sqrt{(r^2 - \omega^2)}]$ both will be negative. Hence the displacement y of the particle decreases exponentially with time t.

Such a motion is called dead-beat or aperiodic and shown in given figure (curve I).

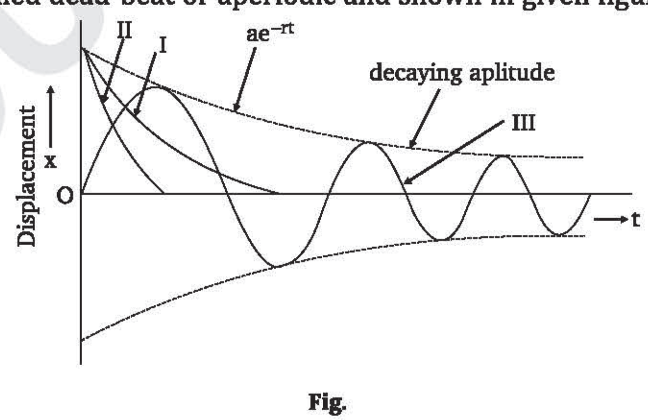


Fig.

II. Critical damping : $r \approx \omega$

Let $\sqrt{r^2 - \omega^2} = h$, where h is a small positive number.

Therefore, eqn. (4) can be written as

$$x = A_1 e^{(-r+h)t} + A_2 e^{(-r-h)t} = e^{-rt}[A_1 e^{ht} + A_2 e^{-ht}]$$

$$= e^{-rt}[A_1(1+ht+\ldots) + A_2(1-ht+\ldots)]$$

$$= e^{-rt}[(A_1+A_2) + ht(A_1-A_2)] \quad \text{[neglecting higher powers of } h\text{]}$$

$$x = e^{-rt}(M+Nt) \qquad \ldots(5)$$

where $M = A_1 + A_2$ and $N = (A_1 - A_2)h$

It is clear from eqn. (5) that as t increases, the value of term $(M+Nt)$ increases but the value of e^{-rt} reduces. Hence the displacement y of the particle under critical damping, first increases due to term $(M+Nt)$ but at once reduces to zero due to e^{-rt} more rapidly compared to case I. This is shown in the given figure (curve II).

III. Under damping : $r < \omega$

If $r < \omega$, then $\sqrt{(r^2 - \omega^2)}$ will be imaginary and we can write it as

$$\sqrt{(r^2 - \omega^2)} = i\sqrt{(\omega^2 - r^2)} = i\beta$$

Putting this value in eqn. (4), we get

$$x = A_1 e^{(-r+i\beta)t} + A_2 e^{(-r-i\beta)t}$$

or $$x = e^{-rt}[A_1 e^{i\beta t} + A_2 e^{-i\beta t}]$$

$$= e^{-rt}[A_1(\cos\beta t + i\sin\beta t) + A_2(\cos\beta t - i\sin\beta t)]$$

$$= e^{-rt}[(A_1+A_2)\cos\beta t + i(A_1-A_2)\sin\beta t] \qquad \ldots(6)$$

Let $A_1 + A_2 = a\sin\phi$ and $i(A_1 - A_2) = a\cos\phi$, then

$$x = e^{-rt}[a\sin\phi\cos\beta t + a\cos\phi\sin\beta t]$$

or $$x = ae^{-rt}\sin(\beta t + \phi)$$

or $$x = ae^{-rt}\sin[\sqrt{(\omega^2 - r^2)}\,t + \phi] \qquad \ldots(7)$$

This is the displacement of a damped harmonic oscillator at time t.

At a time t: Amplitude $= ae^{-rt} = ae^{-(bt/2m)}$

and angular frequency $$\beta = \sqrt{(\omega^2 - r^2)} = \sqrt{\left[\frac{k}{m} - \left(\frac{b}{2m}\right)^2\right]}$$

Time period, $$T = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\left(\frac{k}{m} - \left(\frac{b}{2m}\right)^2\right)}}$$

Q.7. What do you mean by forced oscillations? Write down and solve the differential equation for a particle executing forced harmonic oscillations.

Ans.

Forced or Driven Harmonic Oscillator

When a harmonic oscillator oscillates in a medium like air, its amplitude falls exponentially with time to zero. If we apply an external periodic force having different frequency than the natural frequency of the oscillator, it starts oscillating with the frequency of applied external force after some time.

An oscillator, thus compelled to oscillate with a frequency other than its own natural frequency, is called a driven harmonic oscillator. Its oscillations are called **driven** or **forced oscillations**.

Let a particle of mass m oscillates about its mean position under the effect of force constant k. Let b be the damping constant and an external periodic force $F_0 \sin pt$ is applied to the particle, then at any instant the following forces act on it :

(i) restoring force $= -kx$

(ii) damping force $= -b\,(dx/dt)$

(iii) external periodic force $= F_0 \sin pt$

Therefore,
$$F = F_0 \sin pt - b\frac{dx}{dt} - kx$$

or
$$m\frac{d^2x}{dt^2} = F_0 \sin pt - b\frac{dx}{dt} - kx$$

or
$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\sin pt \qquad ...(1)$$

Let, $\frac{b}{m} = 2r, \frac{k}{m} = \omega_0^2$ and $f_0 = \frac{F_0}{m}$, then
$$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega_0^2 x = f_0 \sin pt \qquad ...(2)$$

This is the differential equation of a forced harmonic oscillator. Its solution has two types.

1. Type I-Transient state solution : This solution shows the transient state of a oscillator, which is the state before the position when the oscillator oscillates with the frequency of applied external force. This is given by the equation.
$$\frac{d^2x}{dt} + 2r\frac{dx}{dt} + \omega_0^2 x = 0$$

This is the equation of damped harmonic oscillator, whose solution is
$$x = ae^{-rt}\sin(\omega t + \phi)$$

2. Type II-Steady state solution : When the transient state is finished, the oscillator starts oscillating with the frequency of applied force. This is governed by eqn (2). Let its solution be
$$x = A\sin(pt - \phi) \qquad ...(3)$$
where A and ϕ are constants and p is the frequency of impressed force.

Now, $$\frac{dx}{dt} = pA\cos(pt - \phi)$$

and $$\frac{d^2x}{dt^2} = -p^2 A\sin(pt - \phi)$$

On putting these values in eqn. (2), we get

$$-p^2 A\sin(pt-\phi) + 2rp\,A\cos(pt-\phi) + \omega_0^2 A\sin(pt-\phi) = f_0 \sin[(pt-\phi)+\phi]$$

$$= f_0 \sin(pt-\phi)\cos\phi + f_0\cos(pt-\phi)\sin\phi$$

or $$A(\omega_0^2 - p^2)\sin(pt-\phi) + 2rp\,A\cos(pt-\phi)$$

$$= f_0\sin(pt-\phi)\cos\phi + \cos(pt-\phi)\sin\phi \quad ...(4)$$

This eqn. holds only if the coefficients of $\sin(pt-\phi)$ and $\cos(pt-\phi)$ are equal on both sides, then we have

$$A(\omega_0^2 - p^2) = f_0\cos\phi \quad ...(5)$$

$$2rpA = f_0 \sin\phi \quad ...(6)$$

On squaring and adding these two eqns., we get

$$A^2[(\omega_0^2 - p^2)^2 + 4r^2p^2] = f_0^2[\cos^2\phi + \sin^2\phi] = f_0^2$$

Hence amplitude, $$A = \frac{f_0}{[(\omega_0^2 - p^2)^2 + 4r^2p^2]^{1/2}} \quad ...(7)$$

On dividing eqn. (6) from eqn. (5), we have

$$\tan\phi = \frac{2rp}{(\omega_0^2 - p^2)} \quad ...(8)$$

Therefore, from eqn. (3), we have

$$x = \frac{f_0}{[(\omega_0^2 - p^2)^2 + 4r^2p^2]^{1/2}} \sin(pt-\phi) \quad ...(9)$$

This is the solution of the forced harmonic oscillator. It gives the displacement of the forced oscillator at any time t.

Q.8. What is wave motion? Write their types in detail.

Ans.

Wave Motion

A periodic motion communicated from particle to particle through an elastic medium is termed as 'wave motion'. Therefore, wave can be defined as the continuous transfer of a state from one part of the medium to another with finite velocity. The medium does not move itself, but the state is propagated through it.

Types of Waves

There are two distinct type of waves-mechanical waves and electromagnetic waves.

1. Mechanical waves

A medium is needed for the production and propagation of mechanical waves. The medium should possess the property of **elasticity** and **inertia**. Sound waves, water waves etc. are their examples.

Mechanical waves are of two types :

(i) **Transverse waves :** In a transverse wave, the particles of the medium execute simple harmonic motion about their mean positions of rest in a direction at right angles to the direction of propagation of the wave *e.g.* waves on a stretched string. There should be properties of **elasticity of shape** or **rigidity** in the medium. Therefore transverse waves are possible in solids as well as on the **surface of liquids**. These waves are not possible in gases. The distance between two consecutive crests or trough is called wavelength of the wave.

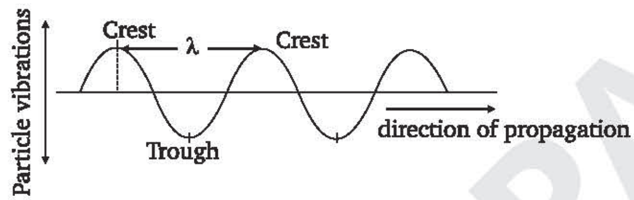


Fig. 1

(ii) **Longitudinal waves :** In a longitudinal wave the particles of the medium execute simple harmonic motion about their mean positions in the direction of propagation of wave *e.g.*, **sound wave in air** and **waves in a stretched spring** etc. These move forward in the form of condensation and rarefaction in the mediums as shown in fig. 2. These are possible in solids, liquids as well as gaseous medium. The distance between two consecutive condensation or rarefactions is the wavelength of the wave.

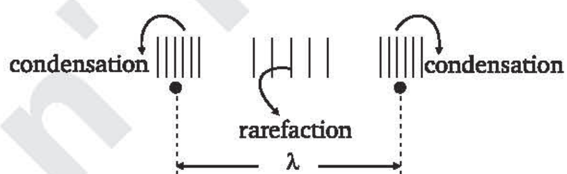


Fig. 2

2. Electromagnetic waves

For the transfer of energy through any medium by means of waves the medium must be elastic, must have the property of inertia and damping should be small. Electromagnetic waves are transverse waves in which electric and magnetic field vectors oscillate perpendicular to the direction of propagation and also are at right angles to each other *e.g.* **light waves, radio waves** etc.

Waves are also classified as one-dimensional (1*D*) two-dimensional (2*D*), three-dimensional (3*D*) according to the directions of transfer of energy. For example

$1D \rightarrow$ waves on a spring or on a stretched string

$2D \rightarrow$ water waves

$3D \rightarrow$ sound waves, light waves etc.

Q.9. Derive the equation for a plane progressive hormonic wave and discuss its properties.

Ans.

Equation of Plane Progressive Wave

In a simple harmonic progressive wave, the particles of the medium also execute simple harmonic motion about their mean positions. If the amplitude of the progressive wave remains unaltered, then it is called a plane progressive wave. Let the particle *O* executes SHM (Fig.), then each particle of the medium will execute similar SHM. If we measure the time from the instant when the particle O passes through its mean position in the positive direction, then the equation of motion of the particle O is given by

$$y = a \sin \omega t \quad ...(1)$$

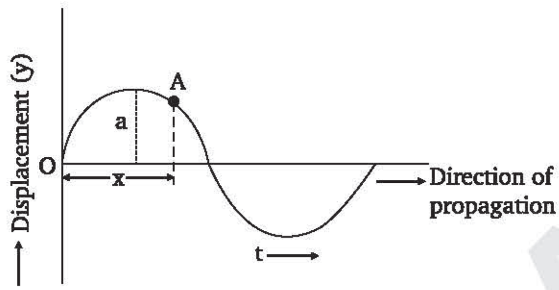


Fig.

where a is the amplitude of the particle, y is the displacement after time t and ω is the angular velocity. If n is the frequency of the vibration, then

$$y = a \sin 2\pi\, nt \quad ...(2)$$

Let A be any particle at a distance x from the origin and the wave be travelling with velocity v along the positive x-direction. Then the particle
A will start vibrating (x/v) seconds after the first particle O starts vibrating, *i.e.* particle A lags behind particle O in the phase by (x/v) seconds.
If ϕ is the phase lag of particle A, then for it

$$y = a \sin (2\pi nt - \phi) \quad ...(3)$$

If the two particles are separated by a distance x, the phase difference between them $= \dfrac{2\pi x}{\lambda} = \phi$.

Putting the value of ϕ in the equation (3), we have

$$y = a \sin \left(2\pi nt - \frac{2\pi x}{\lambda}\right) \quad ...(4)$$

$$= a \sin 2\pi \left(nt - \frac{x}{\lambda}\right)$$

$$= a \sin 2\pi \left(\frac{vt}{\lambda} - \frac{x}{\lambda}\right) \quad [\because v = n\lambda]$$

$$\therefore \quad y = a \sin \frac{2\pi}{\lambda}(vt - x) \quad ...(5)$$

This is general equation of a plane progressive wave of amplitude a propagating with velocity v along the positive direction of x-axis.
If the wave is travelling in the negative direction of x-axis, then

$$y = a \sin \frac{2\pi}{\lambda}(vt + x) \quad ...(6)$$

Since $\omega = 2\pi n$ and propagation constant $k = \dfrac{2\pi}{\lambda}$, equation (4) can also be written as

$$y = a \sin (\omega t - kx) \quad ...(7)$$

Since $\omega = 2\pi n = \dfrac{2\pi}{T}$, equation (4) also becomes

$$y = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda} \right) \qquad ...(8)$$

On differentiating eqn. (5) with respect to t, we get particle velocity as

$$u = \frac{dy}{dt} = a \frac{2\pi v}{\lambda} \cos (vt - x)$$

Maximum particle velocity, $u_{max} = \dfrac{2\pi\, av}{\lambda}$

The acceleration f of any particle at any time in its vibratory motion is given by

$$f = \frac{du}{dt} = \frac{d}{dt} \left[\frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \right]$$

or

$$f = -\left(\frac{2\pi v}{\lambda} \right)^2 a \sin \frac{2\pi}{\lambda} (vt - x)$$

or

$$f = -\frac{4\pi^2 v^2 y}{\lambda^2}$$

Maximum value of acceleration $= f_{max} = \dfrac{4\pi^2 v^2 a}{\lambda^2}$ $\qquad \because y_{max} = a$

Q.10. Define energy of a progressive wave and also explain their equation.

Ans.

Energy of a Progressive Wave

When a progressive wave moves in any medium, then particles of the medium also start vibrating. For this they get energy from the wave and wave compensates this loss in wave energy by absorbing energy from the source.

We have

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \qquad ...(1)$$

Velocity of particle is given by

$$u = \frac{dy}{dt} = \frac{2\pi v}{\lambda} a \cos \frac{2\pi}{\lambda} (vt - x) \qquad ...(2)$$

Acceleration of particle is given by

$$f = \frac{d^2 y}{dt^2} = -\left(\frac{2\pi v}{\lambda} \right)^2 a \sin \frac{2\pi}{\lambda} (vt - x) \qquad ...(3)$$

If ρ is the density of medium, then mass of its unit volume will be ρ and the mass of particles present in unit volume will also be ρ because medium is composed of these particles.

Therefore, kinetic energy of the particles present in unit volume is given by

$$E_K = \frac{1}{2} \rho \left(\frac{dy}{dt} \right)^2$$

or

$$E_K = \frac{1}{2} \rho \frac{4\pi^2 v^2 a^2}{\lambda^2} \cos^2 \frac{2\pi}{\lambda} (vt - x) \qquad ...(4)$$

Now, work done to give a displacement dy to particle of unit mass

$$= (\text{force on unit mass}) \times dy$$

$$= \left(\rho \times \frac{d^2 y}{dt^2}\right) \times dy$$

$$= \rho\left[-\left(\frac{2\pi v}{\lambda}\right)^2 a \sin\frac{2\pi}{\lambda}(vt - x)\right] dy$$

$$= -\rho\,\frac{4\pi^2 v^2}{\lambda^2}\, y\, dy$$

$$= \text{potential energy of unit volume.}$$

Therefore for complete displacement y, potential energy in unit volume

$$E_P = \int \rho\,\frac{4\pi^2 v^2}{\lambda^2}\, y\, dy = \rho\,\frac{4\pi^2 v^2}{\lambda^2}\cdot\frac{y^2}{2}$$

$$= \frac{1}{2}\rho\,\frac{4\pi^2 v^2}{\lambda^2} a^2 \sin^2\frac{2\pi}{\lambda}(vt - x) \qquad \text{...(5)}$$

Hence, total energy of unit volume of medium

$$E = E_K + E_P$$

$$E = \left[\frac{1}{2}\rho\,\frac{4\pi^2 v^2}{\lambda^2} a^2 \cos^2\frac{2\pi}{\lambda}(vt - x)\right] + \left[\frac{1}{2}\rho\,\frac{4\pi^2 v^2}{\lambda^2} a^2 \sin^2\frac{2\pi}{\lambda}(vt - x)\right]$$

$$E = \frac{1}{2}\rho\,\frac{4\pi^2 v^2}{\lambda^2} a^2 \left[\sin^2\frac{2\pi}{\lambda}(vt - x) + \cos^2\frac{2\pi}{\lambda}(vt - x)\right]$$

or $$E = \frac{1}{2}\rho\,\frac{4\pi^2 v^2}{\lambda^2} a^2 \qquad \text{...(6)}$$

or $$E = \frac{1}{2}\rho 4\pi^2\,\frac{n^2\lambda^2 a^2}{\lambda^2} \qquad [\because v = n\lambda]$$

or $$E = 2\pi^2 n^2 a^2 \rho \qquad \text{...(7)}$$

This is the total energy of unit volume of the medium.

It is clear that $E \propto n^2$ and $E \propto a^2$

Thus total energy of the wave is proportional to the **square of its frequency** and also to the **square of its amplitude**.

Energy flowing per unit area per sec is called the intensity I of the wave.

Thus $$I = \frac{\text{energy}}{\text{area} \times \text{time}} = \frac{\text{energy} \times \text{lenfth}}{\text{volume} \times \text{time}} = \left(\frac{\text{energy}}{\text{volume}}\right)\left(\frac{\text{length}}{\text{time}}\right)$$

$$I = Ev = (2\pi^2 n^2 a^2 \rho)\, v$$

Q.11. Explain the wave velocity and group velocity. Also obtain a relation between them.

Ans.

Wave Velocity and Group Velocity

The equation of a plane progressive harmonic wave, travelling along *x*-axis is

$$y = a \sin(\omega t - kx)$$

where $\omega = 2\pi n$, is the angular frequency of the wave and $k = \frac{2\pi}{\lambda}$ is the wave vector.

For a wave, the ratio of ω and k is called the **phase velocity** or **wave velocity** and is denoted by u_p.

$$\therefore \qquad v_p = \frac{\omega}{k}$$

Obviously, this is the velocity of a plane wavefront or plane of constant phase, given by $(\omega t - kx) = \text{constant}$. Now on differentiating w.r.t time we get

$$\omega - k\frac{dx}{dt} = 0 \quad \text{or} \quad \frac{dx}{dt} = v_p = \frac{\omega}{k} \qquad \text{...(1)}$$

Let us consider a group of progressive waves having frequencies in the range ω and $\omega + \Delta\omega$ and wave vectors k and $k + \Delta k$, superimposing together. At some point in space, all the constituent waves will superimpose in phase to produce maximum amplitude, while at other points the different component waves partially or totally will be cancelling mutually their effect. Then around the point of reinforcement, a **wave packet** will be formed in a finite region of space. Let us consider the superposition of two waves travelling along the *x*-axis, given by

$$y_1 = a \sin(\omega_1 t - k_1 x)$$

and

$$y_2 = a \sin(\omega_2 t - k_2 x)$$

where, $k_1 = \frac{2\pi}{\lambda_1}, k_2 = \frac{2\pi}{\lambda_2}$ and $\frac{\omega_1}{k_1} = v_1$ and $\frac{\omega_2}{k_2} = v_2$.

The equation of the resultant wave is given by

$$y = y_1 + y_2$$

$$y = a \sin(\omega_1 t - k_1 x) \; a \sin(\omega_2 t - k_2 x)$$

$$y = 2a \sin\left[\frac{(\omega_1 t - k_1 x) + (\omega_2 t - k_2 x)}{2}\right] \times \cos\left[\frac{(\omega_1 t - k_1 x) - (\omega_2 t - k_2 x)}{2}\right]$$

$$y = 2a \cos\left[\frac{(\omega_1 - \omega_2)t}{2} - \frac{(k_1 - k_2)x}{2}\right] \times \sin\left[\frac{(\omega_1 + \omega_2)t}{2} - \frac{(k_1 + k_2)x}{2}\right]$$

Since $\omega_1 - \omega_2 = \Delta\omega$ and $k_1 - k_2 = \Delta k$

and $\qquad \frac{\omega_1 + \omega_2}{2} \cong \omega$ and $\frac{k_1 + k_2}{2} \cong k,$

Then above equation reduces to

$$y = \left[2a \cos\frac{(\Delta\omega t - \Delta kx)}{2}\right] \sin(\omega t - kx)$$

This equation represents a wave packet of angular velocity ω, whose amplitude $\left[2a\cos\frac{\Delta\omega t - \Delta kx}{2}\right]$ varies with time t and space x and whose maximum value will be $2a$. The velocity of this maximum amplitude or velocity of movement of the packet is called the group velocity.

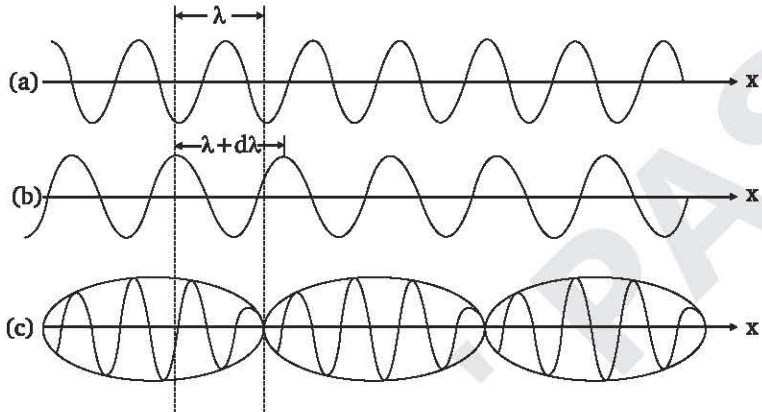


Fig.

Hence, group velocity, $v_g = \frac{\Delta\omega/2}{\Delta k/2} = \frac{\Delta\omega}{\Delta k}$

If difference between frequencies is very small, then

$$v_g = \frac{d\omega}{dk}$$

Relation between Group Velocity and Phase Velocity : The phase velocity of a wave is

$$v_p = \frac{\omega}{k} \text{ or } \omega = kv_p$$

The group velocity is $v_g = \frac{d\omega}{dk} = \frac{d}{dk}(kv_p) = v_p + k\frac{dv_p}{dk}$

But $k = \frac{2\pi}{\lambda}$

Hence $dk = d\left(\frac{2\pi}{\lambda}\right) = -\frac{2\pi}{\lambda^2}d\lambda$

Therefore, $v_g = v_p + \frac{2\pi}{\lambda}\left[\frac{dv_p}{\left(\frac{-2\pi}{\lambda^2}\right)d\lambda}\right]$ or $v_g = v_p - \lambda\frac{dv_p}{d\lambda}$ ...(1)

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MODEL PAPER

Mathematical Physics and Newtonian Mechanics

B.Sc.-I (SEM-I) [M.M. : 75

Note : Attempt all the sections as per instructions.

Section-A : Very Short Answer Type Questions

Instruction : Attempt all **FIVE** questions. Each question carries **3 Marks.** Very Short Answer is required, not exceeding 75 words.

1. Position vector of a particle is $\vec{r}(t) = 4t^2\hat{i} + 5t\hat{j} - 3\hat{k}$ meter, where t is measured in second. Calculate the displacement of particle in time interval $t = 1s$ and $t = 4s$.
2. What are solenoidal field and irrotational field?
3. What is meant by a coordinate system?
4. What do you mean by gravitational force?
5. What is meant by amplitude?

Section-B : Short Answer Type Questions

Instruction : Attempt all **TWO** questions out of the following 3 questions. Each question carries **7.5 Marks.** Short Answer is required not exceeding 200 words.

6. Define a pseudo vector. Show that cross product of two polar vector is pseudo vector while the vector triple product of three polar vector is a true vector.

Or Define partial derivatives of a vector.

7. Differentiate between one dimensional and two dimensional coordinate systems.

Or The distance between the centres of the carbon and oxygen atoms in the CO molecule is 1.130×10^{-10} m. Locate the centre of mass of the molecule relative to the carbon atom.

8. What is a central force? Explain with example.

Or A bullet of mass 20 g is fired with a speed of 1000 m/s from a freely hanging gun of mass 2.0 kg. Calculate the recoil velocity of gun.

Section-C : Long Answer Type Questions

Instruction : Attempt all **THREE** questions out of the following 5 questions. Each question carries **15 Marks.** Answer is required in detail, between 500-800 words.

9. What do you mean by scalar product of two vectors? Give its geometrical interpretation. Obtain expression for scalar product of two vectors in terms of their Cartesian components. Mention its important properties and physical significance also.

Or What do you mean by derivative of a vector? Gives its geometrical interpretation. Obtain expression for derivative of a vector in terms of cartesian components.

10. 1 kg, 2 kg and 3 kg masses are placed at three corners of equilateral triangle having each arm 1 meter, calculate the centre of mass of this system.

Or A 500 g mass is whirled round in a circle at the end of a string 50 cm long, the other end of which is held in the hand. If the mass makes 8 rev/s, what is its angular momentum? If the number of revolutions is reduced to just one, after 20 seconds, calculate the mean valueof torque acting on the mass.

11. Differentiate between contravariant and covariant tensors on the basis of transformation laws obeyed by them and hence show that the velocity of the fluid at any point is contravariant tensor of rank-1 and also explain rank-2.

Or What is the effect of centrifugal force on acceleration due to gravity. Show that $g\lambda = g - \omega^2 R\cos^2\lambda$.

12. Derive the expression for the rotational kinetic energy of a body and show that the total kinetic energy of a body of mass M and radius R, rolling without slipping along a plane surface is $\frac{1}{mv}\left(1+\frac{k^2}{R^2}\right)$, where v is the linear velocity of the body and K is its radius of gyration about an axis through its centre of mass.

Or Show that a two particle problem under central force can always be reduced to equivalent one particle problem. What is reduced mass?

13. A satellite of mass m is going round the earth (mass M_e) in a circular orbit of radius R_e. Write down its angular momentum J about the centre of its orbit and express its total energy E in terms of its angular momentum.

Or What do you understand by damped oscillator? Set up the differential equation for damped harmonic motion and solve it to obtain the expression for displacement. Discuss heavy damped critical damped and underdamped cases.

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